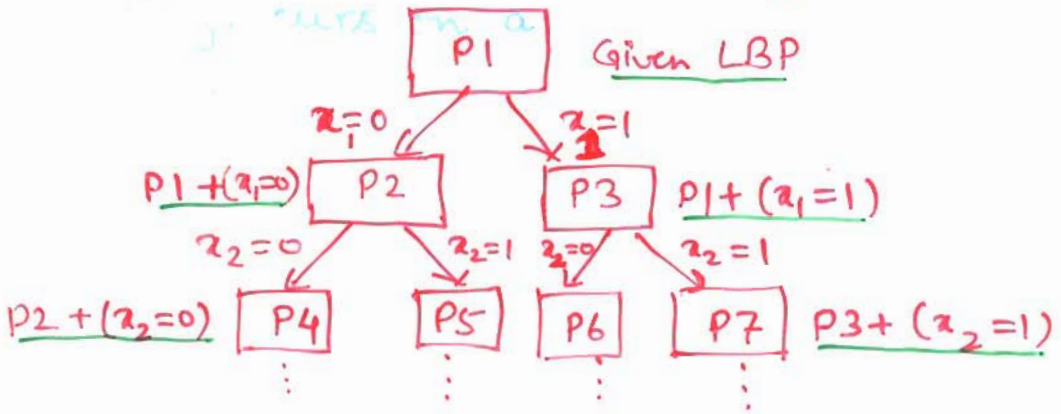
Branch & Bound for Combinatorial Optimization

- Combinatorial Optimization ≡ Decision variables integer & bounded
- Such problems can be converted to BP
  Suppose $x_i < 2^{n+1}$, then replace $x_i = \sum_{k=0}^{n} 2^k q_k^{(i)}$ ($q_k^{(i)} = 0$ or 1
  Example: $x_i < 32 = 2^{4+1} \Rightarrow x_i = 16 q_4^{(i)} + 8 q_3^{(i)} + 4 q_2^{(i)} + 2 q_1^{(i)} + q_0^{(i)}$
- So we study branch & bound for LBP.
  - No need to analyze "LP relaxations", rather analyze "LBPs".
- Root node : Given LBP ; Branching occurs on ($x_i = 0$) or ($x_i = 1$)



- n binary decision variables ⇒ tree can have upto 2^n nodes
- At P1, all variables are "free";
  At P2/P3, x_1 is fixed, others are free;
  At P4/P5/P6/P7, x_1 & x_2 are fixed, others are free.
- At a node "free variables" are available for assignment; they can be assigned for "best completion" (BC) wrt a constraint/objecti

$$\min z = 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5$$

$$\text{s.t.} \quad 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \quad (C1)$$

$$4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7 \quad (C2)$$

- At P6, $x_1 = 1$ & $x_2 = 0$ fixed; x_3, x_4, x_5 "free". BC wrt z: $x_3 = x_4 = x_5$
  BC wrt C1: $x_3 = 0, x_4 = x_5 = 1$; BC wrt C2: $x_3 = 1, x_4 = x_5 = 0$.

Branch & Bound for Comb. Opt.

- At a node branch on a free variable unless

(i) BC wrt any constraint infeasible,

(ii) BC wrt z is inferior than current bound (initial bound $= -\infty$ for max and $+\infty$ for min)

(iii) BC wrt z is also feasible.

(Stop, when no branching possible $\Rightarrow$ Current bound is optimal

Example: (Given LBP)

$$\begin{aligned} \min\ z &= 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\ \text{s.t.}\quad & 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \quad (C1) \\ & 4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7 \quad (C2) \\ & x_i = 0 \text{ or } 1 \end{aligned}$$

Iteration 1: Bound $= \infty$. P1 = Given LBP $\Rightarrow$ all variables free.

(i) BC wrt C1: $x_1 = x_2 = x_4 = x_5 = 1, x_3 = 0 \Rightarrow$ LHS(C1) $= 9 \geq 3$
BC wrt C2: $x_1 = x_2 = x_3 = 1, x_4 = x_5 = 0 \Rightarrow$ LHS(C2) $= 10 \geq 7$ } not infeas

(ii) BC wrt z: $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, which is infeasible (violates C1 & C2)
$z = 0$, which is better than current bound

$\Rightarrow$ branch on a free variable, say x_1

P2 = P1 + $(x_1 = 0)$; P3 = P1 + $(x_1 = 1)$.

Iteration 2: Bound $= \infty$. P2 $\Rightarrow$ $x_1 = 0$ is fixed; x_2, x_3, x_4, x_5 free

(i) BC wrt C1: $x_2 = x_4 = x_5 = 1, x_3 = 0 \Rightarrow$ LHS(C1) $= 5 \geq 3$
BC wrt C2: $x_2 = x_3 = 1, x_4 = x_5 = 0 \Rightarrow$ LHS(C2) $= 6 \not\geq 7$

BC wrt C2 infeasible $\Rightarrow$ no further branching at P2.

Branch & Bound for LBP

Given LBP = P1

$$\begin{cases} \min z = 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\ \text{s.t.} \quad 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \quad (C1) \\ \qquad 4x_4 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7 \quad (C2) \end{cases}$$

Iteration 3: Bound = ∞. P3 $\Rightarrow$ $x_1 = 1$ is fixed; x_2, x_3, x_4, x_5 free

(i) BC wrt C_i = same as P1 (NOT infeasible)

(ii) BC wrt z = $\{x_2 = x_3 = x_4 = x_5 = 0 \Rightarrow$ LHS(C1) $= 4 \geq 3$; LHS(C2) $= 4 \not\geq 7$ $\Rightarrow z = 7$

$\Rightarrow$ branch on a free variable, say x_2

P4 = P3 + $(x_2 = 0)$; P5 = P3 + $(x_2 = 1)$.

Iteration 4: Bound = ∞. P4 $\Rightarrow$ $x_1 = 1$, $x_2 = 0$ fixed; x_3, x_4, x_5 free

(i) BC wrt C1 : $x_3 = 0$, $x_4 = x_5 = 1$ $\Rightarrow$ LHS(C1) $= 7 \geq 3$
BC wrt C2 : $x_3 = 1$, $x_4 = x_5 = 0$ $\Rightarrow$ LHS(C2) $= 8 \geq 7$
} not infeasible

(ii) BC wrt z : $\{x_3 = x_4 = x_5 = 0 \Rightarrow$ LHS(C1) $= 4 \not\geq 3$; LHS(C2) $= 4 \not\geq 7$ $\Rightarrow z = 7$

$\Rightarrow$ branch on a free variable, say x_3.

P6 = P4 + $(x_3 = 0)$; P7 = P4 + $(x_3 = 1)$

Iteration 5: Bound = ∞. P6 $\Rightarrow$ $x_1 = 1$, $x_2 = x_3 = 0$ fixed; x_4, x_5 free

(i) BC wrt C1 : $x_4 = x_5 = 1$ $\Rightarrow$ Same as P4
BC wrt C2 : $x_4 = x_5 = 0$ $\Rightarrow$ LHS(C2) $= 4 \not\geq 7$
} $\Rightarrow$ infeasible.

So no further branching at P6.

Iteration 6: Bound = ∞. P7 $\Rightarrow$ $x_1 = 1$, $x_2 = 0$, $x_3 = 1$ fixed; x_4, x_5 free.

(i) BC wrt C1 : $x_4 = x_5 = 1$ $\Rightarrow$ LHS(C1) $= 6 \geq 3$
BC wrt C2 : $x_4 = x_5 = 0$ $\Rightarrow$ LHS(C2) $= 8 \geq 7$
} not infeasible

(ii) BC wrt z : $\{x_4 = x_5 = 0 \Rightarrow$ LHS(C1) $= 3 \geq 3$; LHS(C2) $= 8 \geq 7 \Rightarrow$ feasible $\Rightarrow z = 9$

$\Rightarrow$ No further branching at P7; new bound = 9

Branch & Bound for LBP

Given LBP = P1

$$\left\{\begin{array}{rl} \min z = & 7x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 \\ \text{s.t.} & 4x_1 + 2x_2 - x_3 + 2x_4 + x_5 \geq 3 \quad (C1) \\ & 4x_1 + 2x_2 + 4x_3 - x_4 - 2x_5 \geq 7 \quad (C2) \end{array}\right\}$$

Iteration 7: Bound = 9. P5 $\Rightarrow$ $x_1 = x_2 = 1$ fixed; x_3, x_4, x_5 free.

(i) BC wrt C1: $x_3 = 0$, $x_4 = x_5 = 1 \Rightarrow$ LHS(C1) $= 4 \geq 3$
BC wrt C2: $x_3 = 1$, $x_4 = x_5 = 0 \Rightarrow$ LHS(C2) $= 10 \geq 7$
} wrt infeasible

(ii) BC wrt z: $x_3 = x_4 = x_5 = 0 \Rightarrow$ { LHS(C1) $= 6 \geq 3$; LHS(C2) $= 6 \not\geq 7$; $z = 10 \Rightarrow$ inferior to current bound }

$\Rightarrow$ No further branching at P5.

optimal $z = 9$; optimal $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = x_5 = 0$.