

1 Consumer's surplus

Consider a household that consumes “power”, denoted by x , and money, denoted by y . A given bundle (x, y) , provides the household with a level of happiness, or utility given by

$$U(x, y) = v(x) + y$$

where v is an increasing and concave function such that $v(0) = 0$. For example $v(x) = \sqrt{x}$. The derivative v' of v is called the *marginal utility* of power. It represents the rate at which the utility increases due to a small addition of power. It is the consumer's willingness to pay for an additional unit of power. This is so because if he consumes one more unit of power and pays for it v' , then his utility will remain the same. In our example $v'(x) = \frac{1}{2\sqrt{x}}$. The household has an income of m and faces a price of power being p . The household has to find the optimal amount of power (and money) to consume, given the price of power p and its income m . Formally, it need to solve the following constrained maximization problem:

$$\begin{aligned} \max_{x,y} v(x) + y \\ \text{s.t. } y + px = m \end{aligned}$$

Substituting the constraint into the objective function, taking derivatives and equalizing to 0, we get

$$v'(x) = p. \tag{1}$$

That is, *the willingness to pay for each additional unit of power should be equal to the price of power*. The solution to equation (1) gives us the demand for power: the optimal amount of power given its price. One can also read this equation as the *inverse demand* for power. For each possible quantity, x , it gives us the market price of power that would induce the consumer to buy exactly x .

In our example, the consumer's problem is

$$\begin{aligned} \max_{x,y} \sqrt{x} + y \\ \text{s.t. } y + px = m \end{aligned}$$

Substituting the constraint into the objective function, taking derivatives and equalizing to 0, we get

$$\begin{aligned} \frac{1}{2\sqrt{x}} &= p \\ \Rightarrow x(p) &= \frac{1}{4p^2}. \end{aligned}$$

What benefit does the household get from participating in the market? If it participates in the market it consumes $(x(p), m - px(p))$. If it does not

participate it consumes $(0, m)$. The difference in utilities is

$$\begin{aligned}
 CS(p) &= v(x(p)) + m - px(p) - (v(0) - m) \\
 &= v(x(p)) - v(0) - px(p) \\
 &= \int_0^{x(p)} v'(x) dx - px(p).
 \end{aligned}$$

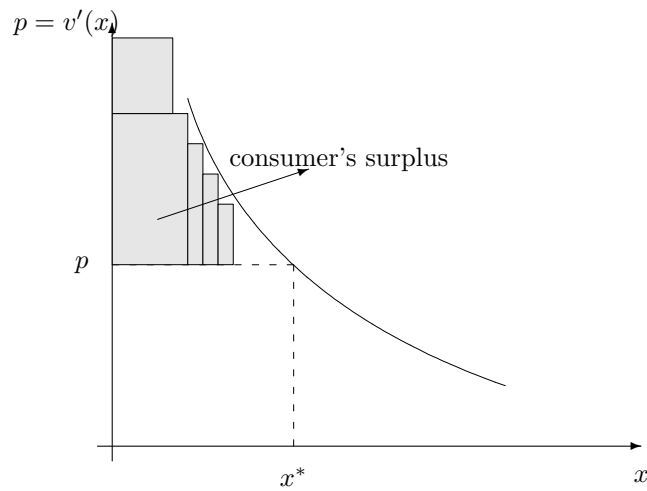


Figure 1: Consumer's surplus.

Suppose the price of power dropped from p_0 to p_1 . What happens to the consumer's surplus?

When the price is p_1 the buyer buys x_1 . When the price is p_0 the buyer buys x_0 . Consequently, the change in utility is

$$\begin{aligned}
 CS(p_1) - CS(p_0) &= U(x_1, m - p_1x_1) - U(x_0, m - p_0x_0) & (2) \\
 &= v(x_1) + m - p_1x_1 - (v(x_0) + m - p_0x_0) \\
 &= v(x_1) - v(x_0) - (p_1x_1 - p_0x_0) \\
 &= (v(x_1) - v(x_0) - p_1(x_1 - x_0)) + ((p_0 - p_1)x_0) \\
 &= \left(\int_{x_0}^{x_1} v'(x) dx - p_1(x_1 - x_0) \right) + ((p_0 - p_1)x_0) & (3) \\
 &= \int_{p_1}^{p_0} x(p) dp.
 \end{aligned}$$

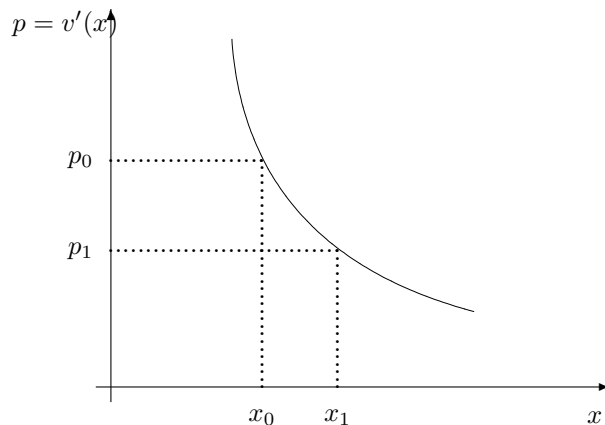


Figure 2: Change in consumer's surplus.

Equation (3) says that the change in the consumer's surplus can be decomposed into two parts: the savings from the reduction in price that we enjoy from buying the old amount, and the increased utility due to the additional purchased amount.

2 Producer's surplus

Consider a generator that produces "power", denoted by q , by means of some technology. Denote the cost of producing q units of power by $C(q)$, where C is assumed to be increasing and convex. The value $C(q)$ is the amount of money that the firm needs to spend in order to produce q units of output. For example, $C(q) = q^2$. The derivative C' of C is called the *marginal cost* of power. It represents the rate at which the cost of production increases due to a small increase in the production of power. In our example $C'(x) = 2q$. The generator faces a price of power being p . The generator has to find the profit maximizing amount of power to produce. Formally, it needs to solve the following constrained maximization problem:

$$\max_{q \geq 0} pq - C(q)$$

Taking derivatives and equalizing to 0, we get

$$C'(q) = p. \quad (4)$$

That is, *the marginal cost of power should be equal to the price of power*. The solution to equation (4) gives us the supply of power: the profit maximizing amount of power given its price. One can also read this equation as the *inverse*

supply for power. For each possible quantity, q , it gives us the market price of power that would induce the generator to sell exactly q .

In our example, the generator's problem is

$$\max_{q \geq 0} pq - q^2$$

Substituting the constraint into the objective function, taking derivatives and equalizing to 0, we get

$$\begin{aligned} 2q &= p \\ \Rightarrow q(p) &= \frac{p}{2}. \end{aligned}$$

What benefit does the generator get from participating in the market? If it participates in the market it produces $q(p)$. If it does not participate it produces 0. The difference in profits is

$$\begin{aligned} pq(p) - C(q(p)) - (0 - C(0)) &= pq(p) - (C(q(p)) - C(0)) \\ &= pq(p) - \int_0^{q(p)} C'(x) dx. \end{aligned}$$

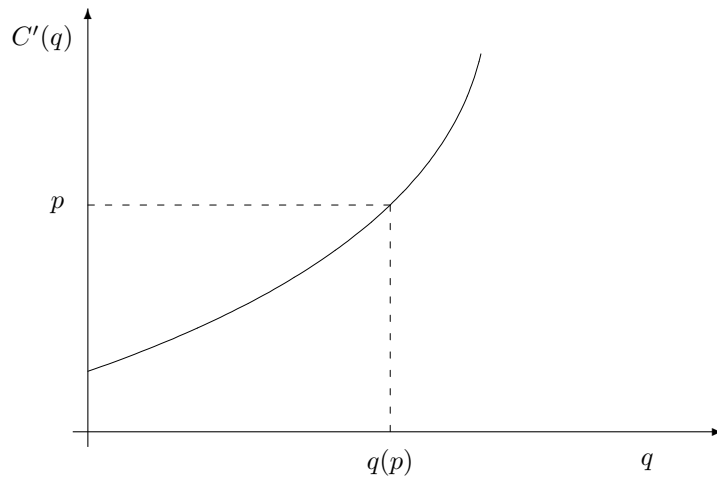


Figure 3: Producer's surplus.

Suppose the price of power increased from p_0 to p_1 . What happens to the producer's surplus?

When the price is p_1 the generator produces q_1 . When the price is p_0 the generator produces q_0 . Consequently, the change in profits is

$$\begin{aligned}
 p_1 q_1 - C(q_1) - (p_0 q_0 - C(q_0)) &= p_1 q_1 - p_0 q_0 - (C(q_1) - C(q_0)) \\
 &= (p_1 - p_0) q_0 + (p_1 (q_1 - q_0) - (C(q_1) - C(q_0))) \\
 &= (p_1 - p_0) q_0 + \left(p_1 (q_1 - q_0) - \int_{q_0}^{q_1} C'(x) dx \right) \quad (5) \\
 &= \int_{p_1}^{p_0} q(p) dp.
 \end{aligned}$$

Equation (5) says that the change in the producer's surplus can be decomposed into two parts: the new revenues from the increase in price that we enjoy from selling the old amount, and the increased profit due to the additional amount sold.

3 Efficiency

Suppose now that we are a benevolent dictator that wants the best for society. The information we have is that the consumer has m units of money, a utility function

$$U(x, y) = v(x) + y$$

and that he owns a the firm that has a cost function $C(q)$. If we want to maximize the happiness in this society, what should we do? What level of output should the firm produce and the consumer consume? To answer this question, consider what would happen if we produced an arbitrary level of output q . The firm would spend $C(q)$ units of money in its production, so that the amount of money left in the economy would be $m - C(q)$. This amount can be distributed between the consumers and the owners of the firm (which we can assume are the consumers themselves). Therefore, if we want to maximize the happiness in this society, we should dictate the production of that level of output that solves

$$\max_{q \geq 0} v(q) + m - C(q).$$

Note that if the consumers pay a price p for each unit of output, the money that remains for them to consume is $m - pq$ while the revenue of the firm is pq . Therefore, the objective function above can be written as

$$(v(q) + m - pq) + (pq - C(q)) = CS(q) + PS(q)$$

Assuming an interior solution to this maximization problem, the first order conditions are

$$v'(q) = C'(q) \quad (6)$$

that is, the marginal utility of the optimal level of output should be equal to its marginal cost. The interpretation is the following. If the willingness to pay for an additional unit of power was higher than its cost of production, we could increase the surplus by producing one more unit. If, on the other hand, the willingness to pay for an additional unit of power was lower than its cost of production, then we could increase the surplus of society by producing one unit less. Only when the willingness to pay for an additional unit of power equals the cost of producing one additional unit can we be maximizing the surplus.

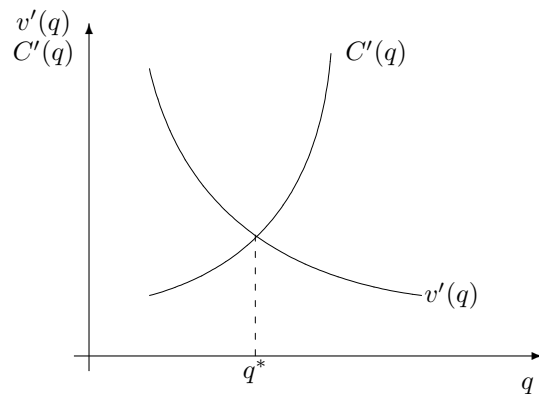


Figure 4: Efficient level of output.

4 Market equilibrium

Unfortunately, there are no benevolent dictators around. Instead, we have a competitive market. In this market, the consumers come and demand power and the generators supply power. The consumers demand power according to their demand functions

$$v'(x) = p$$

while the generators supply power according to their supply functions

$$C'(q) = p.$$

In the market equilibrium, there is an equilibrium price p^* that makes the quantity demanded be equal to the quantity supplied:

$$\begin{aligned} v'(q^*) &= p^* \\ C'(q^*) &= p^*. \end{aligned}$$

It can be seen that the solution to this market equilibrium solves the first order conditions (6). We have shown that the market equilibrium attains the best possible allocation of resources.

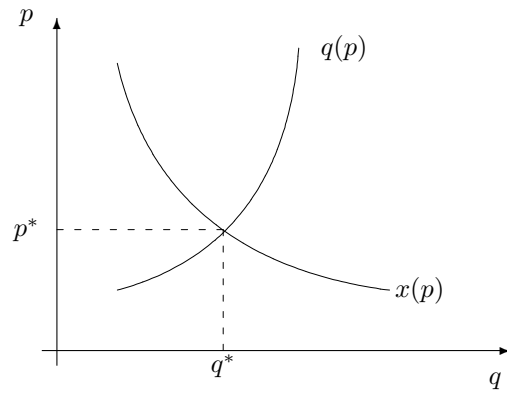


Figure 5: Market equilibrium.