1 The Envelope Theorem

1.1 The unconstrained case

Consider a simple maximization problem

$$\max_x f(x, \theta) \tag{1}$$

where x is a choice variable and θ is a parameter that we do not control. We assume that f is a concave function of x so that the first order conditions are not only necessary but sufficient for maximization. An interior solution of (1) satisfies the following first order conditions:

$$\frac{\partial f}{\partial x}(x, \theta) = 0. \tag{2}$$

The solution to this equation is the maximizer we are looking for. Let's denote it by $x^*(\theta)$. It is clearly a function of θ.

Let's plug this optimal value into the objective function and define the optimal value of the objective function as a function of θ.

$$V(\theta) \equiv f(x^*(\theta), \theta).$$

We are interested in what happens to the optimal value of f when the parameter θ changes. Formally, we are interested in

$$\frac{dV}{d\theta}(\theta).$$

Note that V will change both because θ affects f and because it also affect the optimal choice of x. To answer this question we take derivatives:

$$\begin{aligned} \frac{dV}{d\theta}(\theta) &= \underbrace{\frac{\partial f}{\partial x}(x^*(\theta), \theta)}_{0} \frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) \\ &= \frac{\partial f}{\partial \theta}(x^*(\theta), \theta) \end{aligned}$$

where the second equality follows from (2).

This proves the simple version of the envelope theorem: the total rate of change in the optimal value of the objective function due to a small change in the parameter θ is simply the rate of change in the objective function f evaluated at the optimal value of x.

$$\frac{dV}{d\theta}(\theta) = \frac{\partial f}{\partial \theta}(x^*(\theta), \theta)$$

1.2 The constrained case

Consider a simple maximization problem

$$\begin{aligned} &\max_x f(x,\theta) \\ &\text{s.t } g(x,\theta) \leq 0 \end{aligned} \tag{3}$$

where x is a choice variable and θ is a parameter that we do not control. We assume that f is a concave function of x and that g is a convex function of x so that the first order conditions are not only necessary but sufficient for maximization.

The Lagrangian is the following:

$$L(x,\theta) = f(x,\theta) - \lambda g(x,\theta) \tag{4}$$

An interior solution of (1) where the constraint is binding satisfies the following first order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x}(x,\theta) &= 0 \\ \frac{\partial L}{\partial \theta}(x,\theta) &= 0. \end{aligned}$$

Equivalently

$$\frac{\partial f}{\partial x}(x,\theta) = \lambda \frac{\partial g}{\partial x}(x,\theta) \tag{5}$$

$$g(x,\theta) = 0 \tag{6}$$

The solution to this equation is the maximizer we are looking for. Let's denote it by $x^*(\theta)$. It is clearly a function of θ.

Note that by plugging $x^*(\theta)$ into the constraint function g, we get the following identity (see (6)):

$$g(x^*(\theta),\theta) \equiv 0.$$

Taking derivatives we get

$$\frac{\partial g}{\partial x}(x^*(\theta),\theta)\frac{\partial x^*}{\partial \theta}(\theta) + \frac{\partial g}{\partial \theta}(x^*(\theta),\theta) = 0$$

or

$$\frac{\partial g}{\partial x}(x^*(\theta),\theta)\frac{\partial x^*}{\partial \theta}(\theta) = -\frac{\partial g}{\partial \theta}(x^*(\theta),\theta) \tag{7}$$

Let's plug this optimal value into the objective function and define the optimal value of the objective function as a function of θ.

$$V(\theta) \equiv f(x^*(\theta),\theta).$$

We are interested in what happens to the optimal value of f when the parameter θ changes. Formally, we are interested in

$$\frac{dV}{d\theta}(\theta).$$

Note that V will change both because θ affects f and because it also affect the optimal choice of x. To answer this question we take derivatives:

$$\begin{aligned}
\frac{dV}{d\theta}(\theta) &= \frac{\partial f}{\partial \theta}(x^*(\theta),\theta) + \underbrace{\frac{\partial f}{\partial x}(x^*(\theta),\theta)}_{\lambda \frac{\partial g}{\partial x}(x^*(\theta),\theta)} \frac{\partial x^*}{\partial \theta}(\theta) \\
&= \frac{\partial f}{\partial \theta}(x^*(\theta),\theta) + \underbrace{\lambda \frac{\partial g}{\partial x}(x^*(\theta),\theta) \frac{\partial x^*}{\partial \theta}(\theta)}_{-\lambda \frac{\partial g}{\partial \theta}(x^*(\theta),\theta)} \\
&= \frac{\partial f}{\partial \theta}(x^*(\theta),\theta) - \lambda \frac{\partial g}{\partial \theta}(x^*(\theta),\theta) \\
&= \frac{\partial L}{\partial \theta}(x^*(\theta),\theta)
\end{aligned}$$

where the second equality follows from (5), the third one from (7), and the last one from (4).

This proves the envelope theorem: the total rate of change in the optimal value of the objective function due to a small change in the parameter θ is simply the rate of change in the Lagrangian L evaluated at the optimal value of x.

$$\frac{dV}{d\theta}(\theta) = \frac{\partial L}{\partial \theta}(x^*(\theta),\theta)$$