The tableau method of solving LP we studied in class need some explanation when some of the constraints are not "less than or equal to".

<u>Example</u>:

$$\begin{aligned}
\max \quad & 2x_1 + 3x_2 + 5x_3 & \\
\text{s.t.} \quad & x_1 + x_2 \leq 2 & \text{—(1)} \\
& 4x_1 + x_2 + x_3 \geq 3 & \text{—(2)} \\
& 8x_1 + 2x_2 = 10 & \text{—(3)} \\
& x_i \geq 0 & \text{—(4)}
\end{aligned}$$

Here (1) is "≤", but (2) & 3 are not "≤".

<u>What is special about (2) & (3)?</u>

Let us introduce slack variables to make all constraints (1) –(3) equality constraints:

$$\begin{aligned}
x_1 + x_2 + x_4 &= 2 & \text{— (5)} \\
x_3 + 4x_1 + x_2 - x_5 &= 3 & \text{— (6)} \\
8x_1 + 2x_2 &= 10 & \text{— (7)} \\
x_i &\geq 0 & \text{— (8)}
\end{aligned}$$

So, we introduce slack - variables in (5) and (6). Notice (7) is same as (3) since no slack variable is needed here.

There are a total of 5 variables and 3 constraints. So we can assign 2 of them <u>freely</u>. I.e.,

\# non-basic variables $= 2$; # basic-variables $= 3$.

Try, $x_1 = x_2 = 0$ (our choice for non-basic $= \{x_1, x_2\}$)

$\Rightarrow$ $x_4 = 2$ (from 5)

$x_5 = -3$ (from 6)

[illegible] $= 10$ (from 7)

In general constraints with not "$\leq$" sign need special attention since they create problem when setting decision variables as zero is tried as initial choice of extreme point.

So for each such constraint (that are not "$\leq$" type) we add an auxiliary variable and set the following auxiliary LP:

$$\max \; -(a_1 + a_2)$$

$$\text{s.t.} \quad x_1 + x_2 + x_4 = 2 \qquad \text{— (9)}$$

$$x_3 + 4x_1 + x_2 - x_5 + a_1 = 3 \qquad \text{— (10)}$$

$$8x_1 + 2x_2 + a_2 = 10 \qquad \text{— (11)}$$

$$x_i \geq 0, \; a_i \geq 0 \qquad \text{— (12)}$$

Notice no auxiliary variable needed in (9) (and it is same as (5)). a_1 and a_2 are auxiliary variables.

If the maximum in above equals "zero", then this means at optimum, $a_1 = a_2 = 0$.
(a_1 or $a_2 > 0 \Rightarrow -(a_1+a_2) < 0$, and so not at maximum).
Then at optimum, (9) – (11) are same as (5) – (7), and so if optimum exists for auxiliary LP, then a solution exists for original LP.

<u>Now to find the optimum of auxiliary LP</u>, we can use the same tableau method.

<u>How to pick initial feasible solution?</u>

Total # of variables = 7, # of constraints = 3

$\Rightarrow$ # non-basic variables = 7 − 3 = 4 ; # basic variables = 3.

For auxiliary LP, choose initial basic variables to be variables in last column, i.e.,

basic variables $= \{x_4, a_1, a_2\}$

$\Rightarrow$ non-basic variables $= \{x_1, x_2, x_3, x_5\}$.

By definition, non-basic variables ~~are set to~~ take the value zero, i.e.,

$x_1 = x_2 = x_3 = x_5 = 0$. This implies:

$x_4 = 2$ (from (9))

$a_1 = 3$ (from (10))

$a_2 = 0$ (from (11))

Thus initial feasible solution of auxiliary LP can be obtained by "inspection". (This ~~is~~ was not the case for the original LP.)

Now we need to create a table with $\{x_1, x_2, x_3, x_5\}$ as non-basic variables. So the objective function must have zero coefficient for basic variables $\{x_4, a_1, a_2\}$.

For this, we need to rewrite $z = -(a_1 + a_2)$ as a function of $\{x_1, x_2, x_3, x_5\}$.

Let us write the tableau for auxiliary LP first.

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	RHS
z	0	0	0	0	0	-1	-1	0
(9)	1	$+1$	0	1	0	0	0	2
(10)	4	$+1$	1	0	-1	1	0	3
(11)	8	2	0	0	0	0	1	10

Since we want to make coefficient for $\{x_4, a_1, a_2\}$ zero in first row, add (10) and (11) to row of z.
((10) is row with "a_1", (11) is row with "a_2")

The new table is:

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	RHS
z	12	1	1	0	-1	0	0	13
(9)	1	1	0	1	0	0	0	2
(10)	4	$+1$	1	0	-1	1	0	3
(11)	8	2	0	0	0	0	1	10

Now we are set to solve auxiliary LP. Notice first row has +ve entries $\Rightarrow$ the current solution not optimal.
Current solution has, non-basis $= \{x_1, x_2, x_3, x_5\}$
basis $= \{x_4, a_1, a_2\}$.

From largest +ve coefficient in 1st row, entering basis $= \{x_1\}$
From smallest ratio test, leaving basis variable $= \{a_1\}$.
$\Rightarrow$ At new feasible solution, $x_2 = x_3 = x_5 = a_1 = 0$
new basic variables $= \{x_4, x_1, a_2\}$.

Transforming the table wrt new feasible solution gives,

	x_1	x_2	x_3	x_4	x_5	a_1	a_2	RHS
z	0	-2	-2	0	2	-3	0	4
(13)	0	$3/4$	$-1/4$	1	$1/4$	$-1/4$	0	$5/4$
(14)	1	$+1/4$	$1/4$	0	$-1/4$	$1/4$	0	$3/4$
(15)	0	0	-2	0	2	-2	1	$+4$

Entering basis $= \{x_5\}$, leaving $= \{a_2\}$

$\Rightarrow$ New basis $= \{x_4, x_1, x_5\}$, New non-basis $= \{x_2, x_3, a_1, a_2\}$.

Since new non-basis includes a_1 and $a_2 \Rightarrow a_1 = a_2 = 0$.
So optimum has been found. At this optimum,
$a_1 = a_2 = 0, = x_2 = x_3 = 0$

So, $x_2 = x_3 = 0$ becomes ~~non basis for the~~ initial non-basic variables for original LP:

$$\max \; 2x_1 + 3x_2 + 5x_3$$

s.t.

$$x_1 + x_2 + x_4 = 2 \quad (5)$$
$$4x_1 + x_2 + x_3 - x_5 = 3 \quad (6)$$
$$8x_1 + 2x_2 = 10 \quad (7)$$
$$x_i \geq 0. \quad (8)$$

3 constraints $\Rightarrow$ 3 basic variables
So, 5 (total) $-$ 3 (basis) $= 2$ (non basic)

~~Choose~~ From solution of auxiliary LP, initial non-basic variables are $\boxed{x_2 = x_3 = 0}$

So we need to set the tableau wrt this ~~new~~ non-basis. This means, eliminating the basic variable x_1 from objective fn. ~~Use last equation to get,~~

First write the given eqs in tableau form.

	x_1	x_2	x_3	x_4	x_5	RHS
z	2	3	5	0	0	0
(5)	1	1	0	1	0	2
(6)	4	1	1	0	-1	3
(7)	8	2	0	0	0	10

To make coefficient of basic variable $\{x_1, x_4, x_5\}$ zero in first row, do.
1^{st} row $- \frac{1}{4}$ (last row), then the new z-row becomes:

z	0	$5/2$	$\boxed{5}$	0	0	$-10/4$
(5)						
(6)						
(7)						

Use the other three rows as before to get the initial tableau wrt the non-basis $= \{x_2, x_3\}$
basis $= \{x_1, x_4, x_5\}$

Now you are ready to optimize the tableau by finding new basis variables as discussed in class.

Note since coefficient for x_3 is largest in 1st row, x_3 becomes new entering basis variable, $\boxed{\text{etc.}}$