Nodal prices-III

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The grid

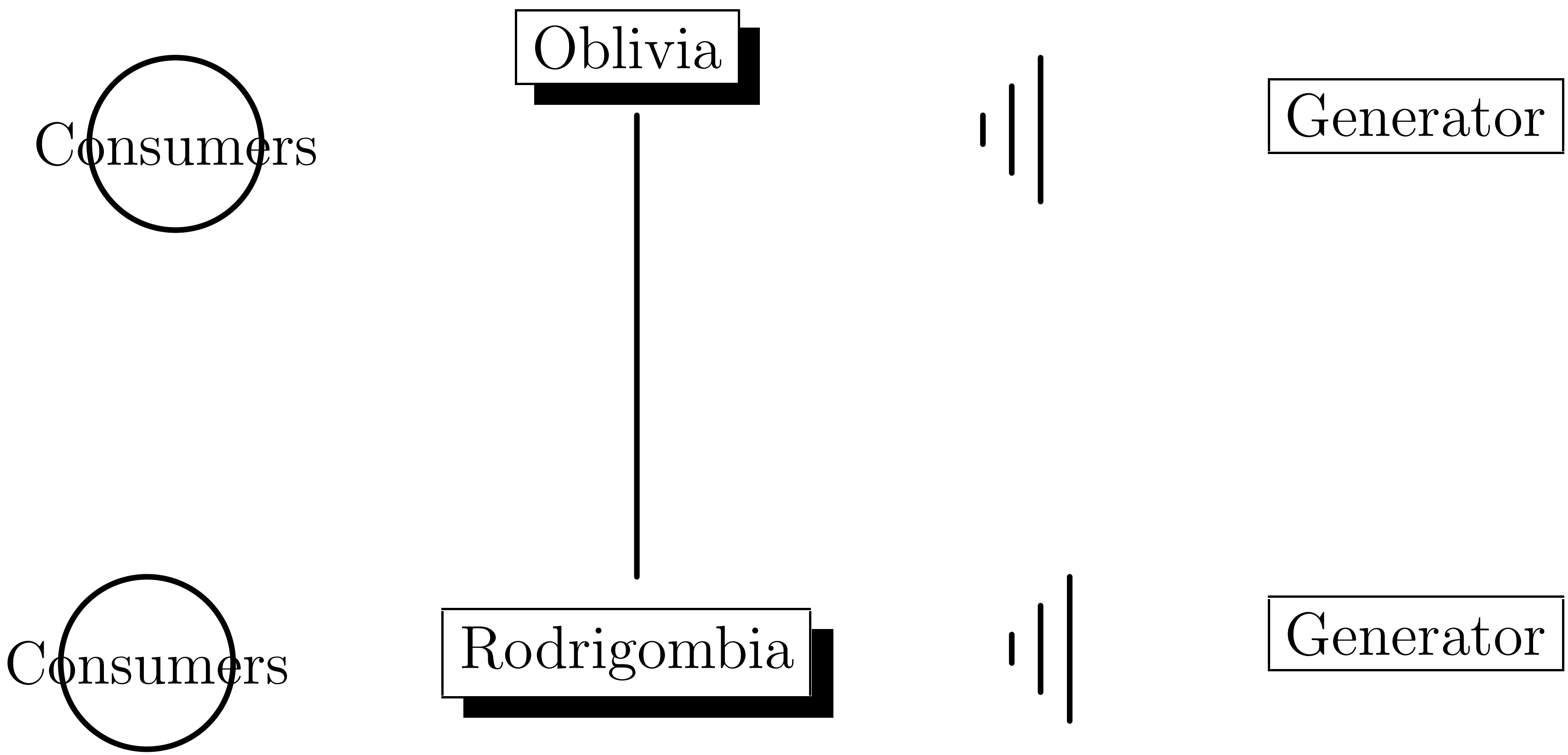

Figure 1: The main actors

Constrained equilibrium

- Assume that the maximum amount of power that can flow through the line is 16 units.
- What is the socially optimal allocation of resources?
- What is the consumption and generation levels in each region that maximize the social surplus?

Social optimum

In order to find the social optimum we need to solve the following problem:

$$\max \quad U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R)$$

$$\text{s.t.} \qquad \begin{cases} x_O + x_R &= q_O + q_R \\ q_O - x_O &\leq 16 \end{cases}$$

In our case, the problem is

$$\max \quad (112 - x_O)\, x_O + 2\, (54 - x_R)\, x_R - \frac{16\, q_O}{5} - \frac{{q_O}^2}{15} - \frac{204\, q_R}{25} - \frac{14\, {q_R}^2}{25}$$

$$\text{s.t.} \qquad \begin{cases} x_O + x_R &= q_O + q_R \\ q_O - x_O &\leq 16 \end{cases}$$

Social optimum

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= (112 - x_O)\, x_O + 2\,(54 - x_R)\, x_R \\ &\quad - \frac{16\, q_O}{5} - \frac{{q_O}^2}{15} - \frac{204\, q_R}{25} - \frac{14\, {q_R}^2}{25} \\ &\quad -\lambda(x_O + x_R - q_O - q_R) - \mu(q_O - x_O - 16)\end{aligned}$$

Social Optimum

The first order conditions are:

$$\begin{aligned}
112 - 2\, x_O - \lambda + \mu &= 0 \\
2\,(54 - x_R) - 2\, x_R - \lambda &= 0 \\
-16/5 - 2\, q_O/15 + \lambda - \mu &= 0 \\
-204/25 - 28\, q_R/25 + \lambda &= 0 \\
q_O + q_R - x_O - x_R &= 0 \\
q_O - x_O - 16 &\leq 0 \\
(q_O - x_O - 16)\mu &= 0
\end{aligned}$$

The solution to this system of equations is

$$\{x_O \to 50, x_R \to 23, q_O \to 66, q_R \to 7, \lambda \to 16, \mu \to 4\}$$

Competitive equilibrium

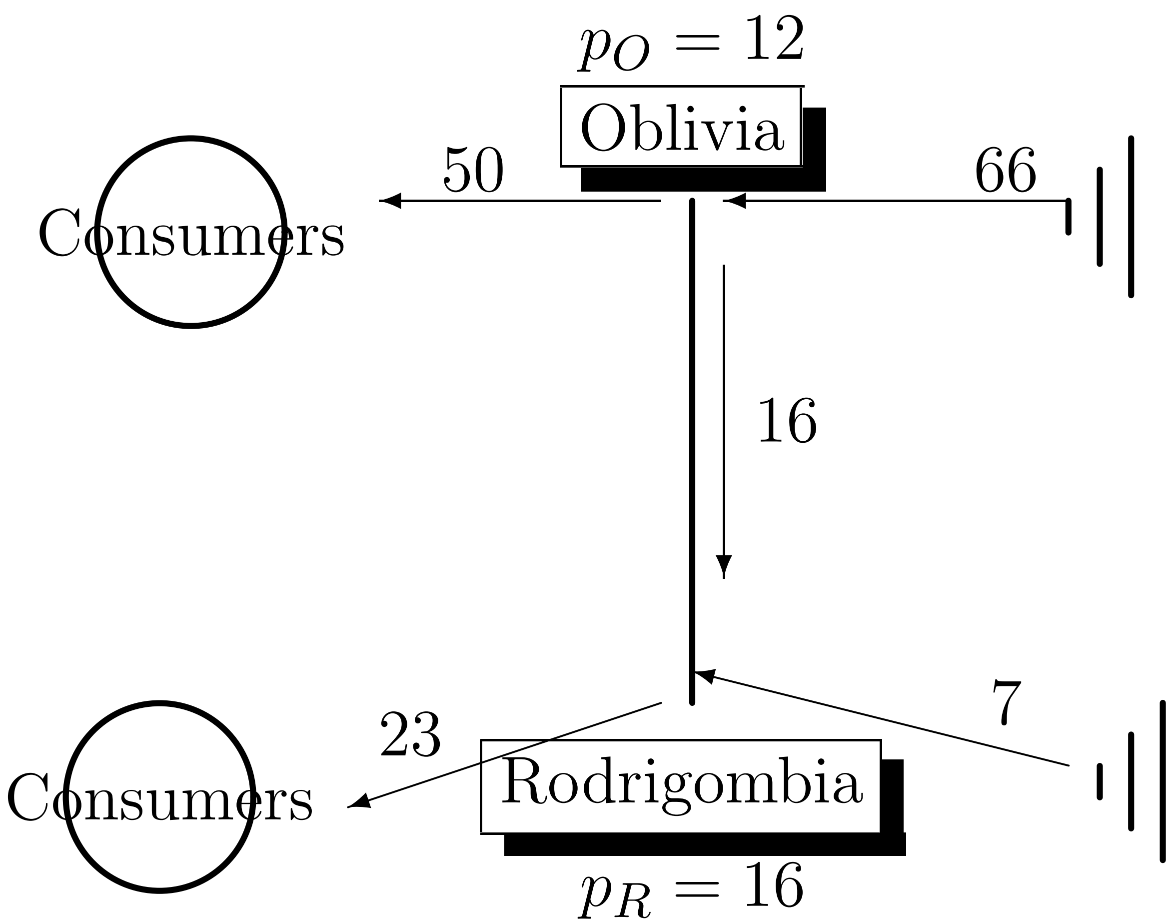

Figure 2: Equilibrium dispatch

Interpretation of λ and μ

In general, a constrained optimal dispatch problem with two buses looks like this:

$$\max \quad U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R)$$

$$\text{s.t.} \quad \begin{cases} x_O + x_R &= q_O + q_R \\ q_O - x_O &\leq K \end{cases}$$

where K is the capacity of the line. (Without loss of generality we assume that bus O is the exporting bus at the optimum).

Social optimum

- The Lagrangian is

$$\begin{aligned}\mathcal{L} &= U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R) \\ &\quad -\lambda(x_O + x_R - q_O - q_R) - \mu(q_O - x_O - K)\end{aligned}$$

Social optimum

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$$\mathcal{L} = U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R)$$
$$-\lambda(x_O + x_R - q_O - q_R) - \mu(q_O - x_O - K)$$

- Denote the solution to this problem by

$$(x_O^*, x_R^*, q_O^*, q_R^*).$$

Social optimum

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- Denote the solution to this problem by

$$(x_O^*, x_R^*, q_O^*, q_R^*).$$

- The value of the social surplus at the optimum is

$$V(K) = U_O(x_O^*) + U_R(x_R^*) - C_O(q_O^*) - C_R(q_R^*).$$

Competitive equilibrium

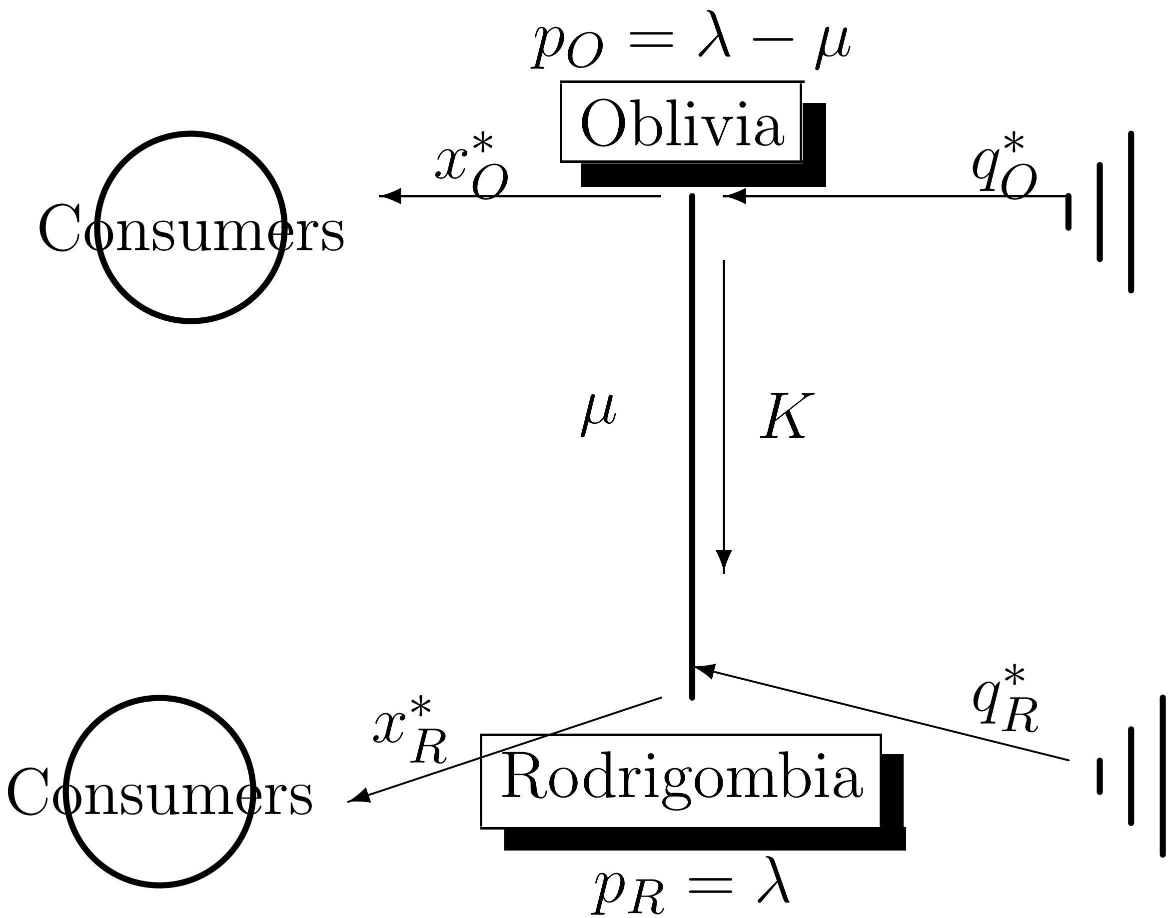

Figure 3: Equilibrium dispatch

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- What happens to the Social Surplus if we have to supply one additional MW to bus R?
- What happens to the Social Surplus if we have to supply one additional MW to bus O?
- What happens to the Social Surplus if we have to transmit one additional MW from bus O to bus R?

The envelope theorem

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$$\frac{dV}{dK}(K) = \frac{\partial \mathcal{L}}{\partial K}(x_O^*, x_R^*, q_O^*, q_R^*, K).$$

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- Therefore,

$$\frac{dV}{dK}(K) = \mu.$$

- The social benefit of an additional unit of transmission capacity is exactly μ.

Notation

In order to answer the other questions we need some notation.

- Denote by θ_O the amount of power that has to be supplied to bus O, in excess of the demand x_O. Think of θ_O as a fixed demand by the Oblivian government. The total load in Oblivia is $(x_O + \theta_O)$.

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- Denote by θ_R the amount of power that has to be supplied to bus R, in excess of the demand x_R. Think of θ_O as a fixed demand by the Rodrigombian government. The total load in Rodrigombia is $(x_R + \theta_R)$.
- Both θ_O and θ_R are 0 in our problem but we can ask ourselves how they would affect the optimal dispatch and social surplus is they were not 0.

The problem again

With the additional notation, the social problem can be stated as follows:

$$\max \quad U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R)$$

$$\text{s.t.} \quad \begin{cases} (x_O + \theta_O) + (x_R + \theta_R) & = & q_O + q_R \\ q_O - (x_O + \theta_O) & \leq & K \end{cases}$$

Social optimum

- The Lagrangian is

$$\begin{aligned}\mathcal{L} &= U_O(x_O) + U_R(x_R) - C_O(q_O) - C_R(q_R) \\ &\quad -\lambda((x_O + \theta_O) + (x_R + \theta_R) - q_O - q_R) \\ &\quad -\mu(q_O - (x_O + \theta_O) - K)\end{aligned}$$

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- Denote the solution to this problem by

$$(x_O^*, x_R^*, q_O^*, q_R^*).$$

- The value of the social surplus at the optimum is

$$V(K, \theta_O, \theta_R) = U_O(x_O^*) + U_R(x_R^*) - C_O(q_O^*) - C_R(q_R^*).$$

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- Answer: By the envelope theorem

$$\begin{aligned}\frac{\partial V}{\partial \theta_R}(K, \theta_O, \theta_R) &= \frac{\partial \mathcal{L}}{\partial \theta_R}(x_O^*, x_R^*, q_O^*, q_R^*, K, \theta_O, \theta_R) \\ &= -\lambda\end{aligned}$$

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- The ***cost*** of supplying an additional MW to the reference bus (bus R) is λ.

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- The **cost** of supplying an additional MW to bus O is $\lambda - \mu$.

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$$\frac{\partial V}{\partial \theta_R}(K, \theta_O, \theta_R) - \frac{\partial V}{\partial \theta_O}(K, \theta_O, \theta_R)?$$

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- What happens to the Social Surplus if we have to transmit one additional MW from bus O to bus R?
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$$\frac{\partial V}{\partial \theta_R}(K, \theta_O, \theta_R) - \frac{\partial V}{\partial \theta_O}(K, \theta_O, \theta_R)?$$

- Answer: By the envelope theorem

$$\begin{aligned} \frac{\partial V}{\partial \theta_R} - \frac{\partial V}{\partial \theta_O} &= \frac{\partial \mathcal{L}}{\partial \theta_O} - \frac{\partial \mathcal{L}}{\partial \theta_R} \\ &= -\mu \end{aligned}$$

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- The **cost** of transmitting an additional MW from bus O to bus R is μ.