
Nodal prices–IV

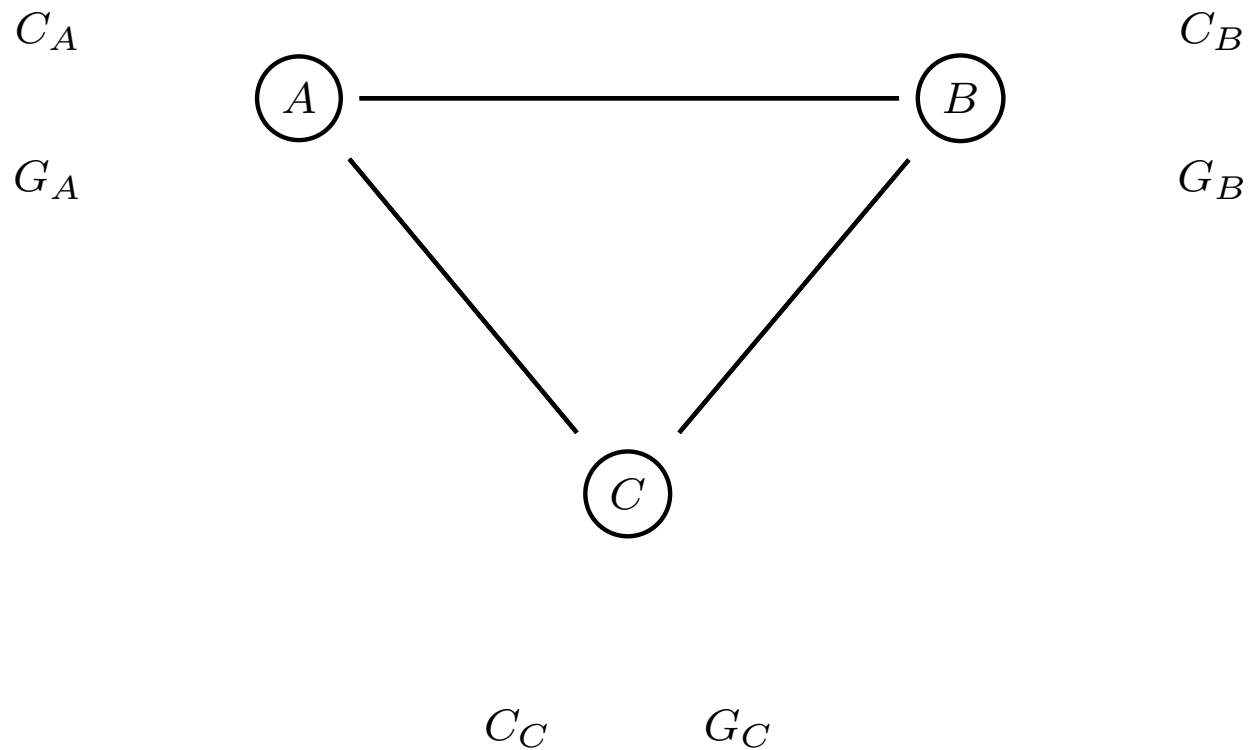
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Motivation

There are three regions: Region A, Region B, and Region C. Each of the three regions has generators and consumers.

The main actors



Region A

The generator in Region A is characterized by the following cost function:

$$C_A(q) = \frac{80q}{21} + \frac{q^2}{2}.$$

The consumers in Region A have the following utility function:

$$U_A(m, x) = m + \left(\frac{2075}{21} - x \right) x$$

Region B

The generator in Region B is characterized by the following cost function:

$$C_B(q) = \frac{67q}{21} + q^2.$$

The consumers in Region B have the following utility function:

$$U_B(m, x) = m + \left(\frac{3301}{21} - x \right) x$$

Region C

The generator in Region C is characterized by the following cost function:

$$C_C(q) = q + 2q^2$$

The consumers in Region C have the following utility function:

$$U_C(m, x) = m + (201 - x)x$$

The Generators Problem

Generators take the price of power p as given, and generate the amount of power that brings their profits to the maximum. Formally, node i 's generator solves

$$\max_{q \geq 0} pq - C_i(q).$$

The solution to this problem is the quantity that solves the first order condition

$$p = \frac{\partial C_i}{\partial q}(q^*).$$

The solution to this equation is a function of the price, and is known as the *generator's supply function*.

Region A's Supply Function

In the case of Region A, the profit maximizing output solves

$$-\left(\frac{80}{21}\right) + p - q = 0$$

Consequently, Region A's generator's supply function is

$$S_A(p) = -\left(\frac{80}{21}\right) + p.$$

Region B's Supply Function

In the case of Region B, the profit maximizing output solves

$$-\left(\frac{67}{21}\right) + p - 2q = 0$$

Consequently, Region B's generator's supply function is

$$S_B(p) = \frac{-67 + 21p}{42}.$$

Region C's Supply Function

In the case of Region C, the profit maximizing output solves

$$-1 + p - 4q = 0$$

Consequently, Region C's generator's supply function is

$$S_C(p) = \frac{-1 + p}{4}.$$

The Consumers' Problem

Consumers take the price of power p and their income I as given, and buy the amount of power that brings their utility to the maximum. Formally, node i 's consumer solves

$$\begin{aligned} \max_{m, q \geq 0} \quad & m + u_i(m, q) \\ \text{s.t.} \quad & m + pq = I. \end{aligned}$$

The solution to this problem is the quantity that solves the first order condition

$$p = \frac{\partial u_i}{\partial q}(q^*).$$

The solution to this equation is a function of the price, and is known as the *consumer's demand function*.

Region A's Demand Function

In the case of Region A, the utility maximizing quantity solves

$$\frac{2075}{21} - p - 2x = 0$$

Consequently, Region A's consumers' demand function is

$$D_A(p) = \frac{2075 - 21p}{42}.$$

Region B's Demand Function

In the case of Region B, the utility maximizing quantity solves

$$\frac{3301}{21} - p - 2x = 0$$

Consequently, Region B's consumers' demand function is

$$D_B(p) = \frac{3301 - 21p}{42}.$$

Region C's Demand Function

In the case of Region C, the utility maximizing quantity solves

$$201 - p - 2x = 0$$

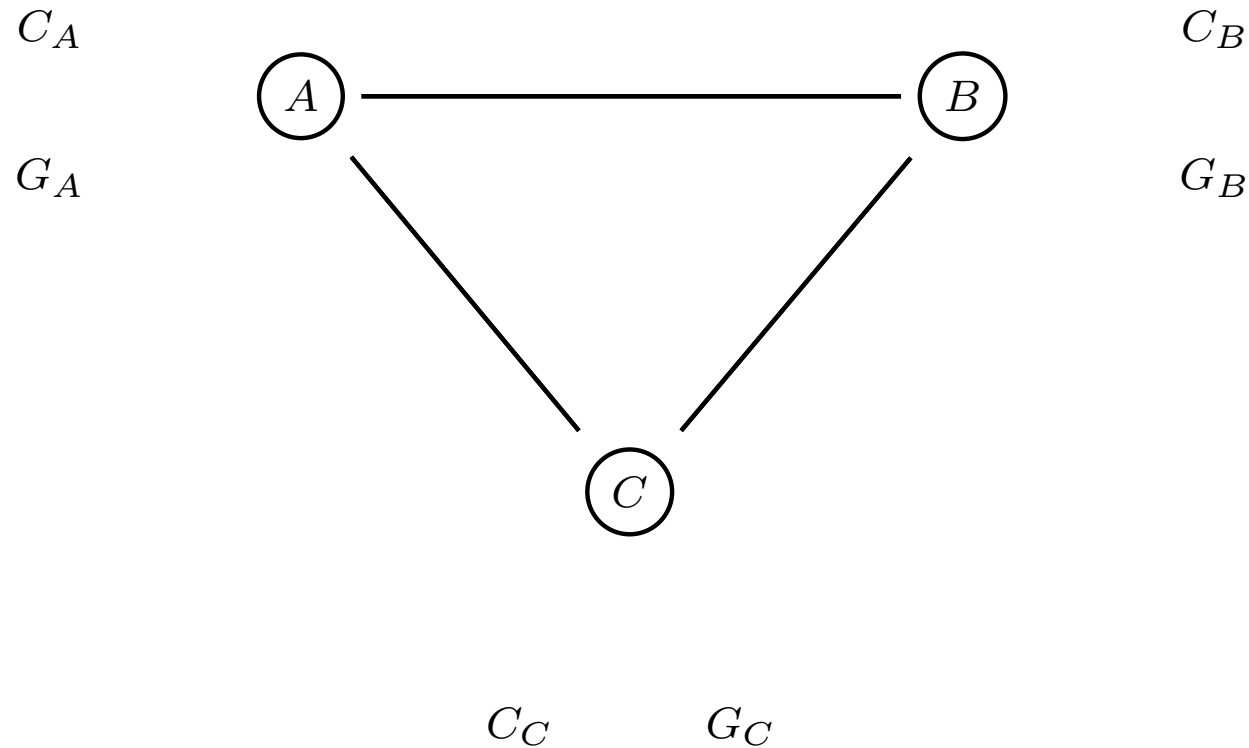
Consequently, Region C's consumers' demand function is

$$D_C(p) = \frac{201 - p}{2}.$$

Unconstrained equilibrium

Assume that the grid that connects Regions A, B, and C has an unlimited amount of power can be transmitted along the lines.

The grid



What is the socially optimal generation and consumption levels at each node?

Social optimum

In order to find the social optimum we need to solve the following problem:

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{R}} (U_i(x_i) - C_i(q_i)) \\ \text{s.t.} \quad & x_A + x_B + x_C = q_A + q_B + q_C \end{aligned}$$

In our case, the problem is

$$\begin{aligned} \max \quad & (m + (\frac{2075}{21} - x) x) - (\frac{80q}{21} + \frac{q^2}{2}) + \\ & (m + (\frac{3301}{21} - x) x) - (\frac{67q}{21} + q^2) + \\ & (m + (201 - x) x) - (q + 2q^2) \\ \text{s.t.} \quad & x_A + x_B + x_C = q_A + q_B + q_C \end{aligned}$$

Social optimum

The Lagrangian is

$$\begin{aligned}\mathcal{L} = & \left(m + \left(\frac{2075}{21} - x\right) x\right) - \left(\frac{80q}{21} + \frac{q^2}{2}\right) + \\ & \left(m + \left(\frac{3301}{21} - x\right) x\right) - \left(\frac{67q}{21} + q^2\right) + \\ & \left(m + (201 - x) x\right) - (q + 2q^2) - \\ & \lambda(x_A + x_B + x_C - q_A - q_B - q_C)\end{aligned}$$

Social Optimum

The first order conditions are:

$$\left\{ \begin{array}{l} \frac{2075}{21} - 2x_A - \lambda = 0 \\ \frac{3301}{21} - 2x_B - \lambda = 0 \\ 201 - 2x_C - \lambda = 0 \\ -\left(\frac{80}{21}\right) - q_A + \lambda = 0 \\ -\left(\frac{67}{21}\right) - 2q_B + \lambda = 0 \\ -1 - 4q_C + \lambda = 0 \\ q_A + q_B + q_C - x_A - x_B - x_C = 0 \end{array} \right.$$

The Economic Dispatch

The solution to this system of equations is

$$x_A \rightarrow \frac{281}{21}, \quad x_B \rightarrow \frac{298}{7}, \quad x_C \rightarrow \frac{1354}{21},$$

$$q_A \rightarrow \frac{1433}{21}, \quad q_B \rightarrow \frac{241}{7}, \quad q_C \rightarrow \frac{373}{21},$$

$$\lambda \rightarrow \frac{1513}{21}$$

Back

Competitive equilibrium

The competitive equilibrium obtains when the market-clearing conditions is satisfied:

- aggregate demand equals aggregate supply.

Aggregate Demand

Aggregate demand:

$$\begin{aligned} D(p) &= D_A(p) + D_B(p) + D_C(p) \\ &= \frac{2075 - 21p}{42} + \frac{3301 - 21p}{42} + \frac{201 - p}{2} \\ &= \frac{457 - 3p}{2}. \end{aligned}$$

Aggregate Supply

Aggregate Supply:

$$\begin{aligned} S(p) &= S_A(p) + S_B(p) + S_C(p) \\ &= -\left(\frac{80}{21}\right) + \frac{-1+p}{4} + p + \frac{-67+21p}{42} \\ &= -\left(\frac{475}{84}\right) + \frac{7p}{4}. \end{aligned}$$

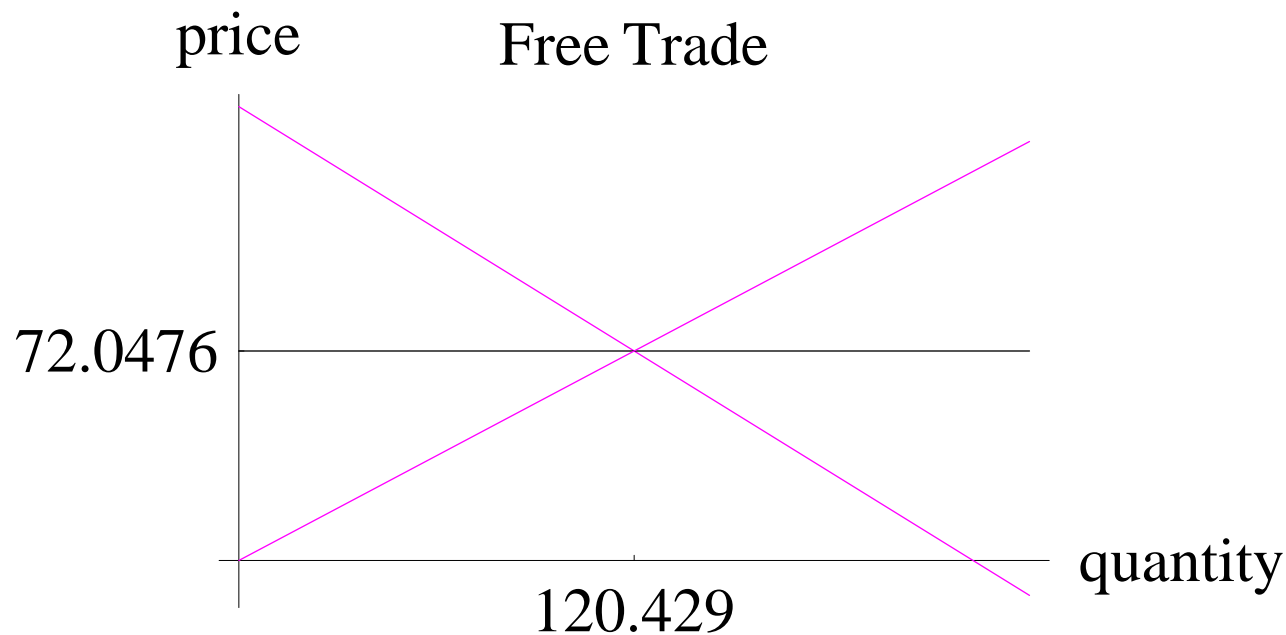
Competitive equilibrium

The market clearing condition is

$$D(p) = S(p)$$

$$\frac{457 - 3p}{2} = -\left(\frac{475}{84}\right) + \frac{7p}{4}.$$

Competitive equilibrium



UNCONSTRAINED MARKET

Competitive equilibrium

The market clearing price is

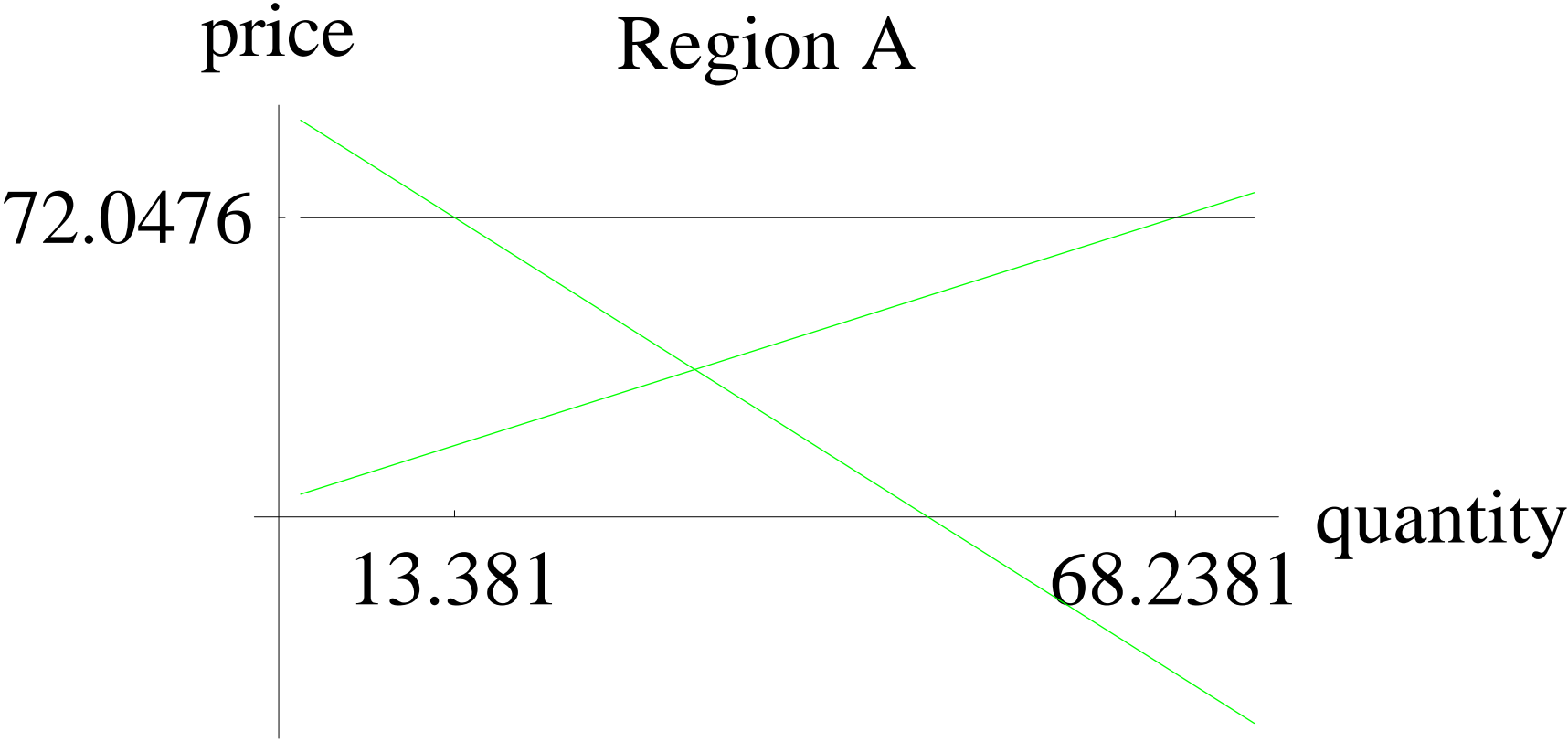
$$p^* = \frac{1513}{21} \approx 72.05$$

and the quantities produced and consumed in each one of the nodes are:

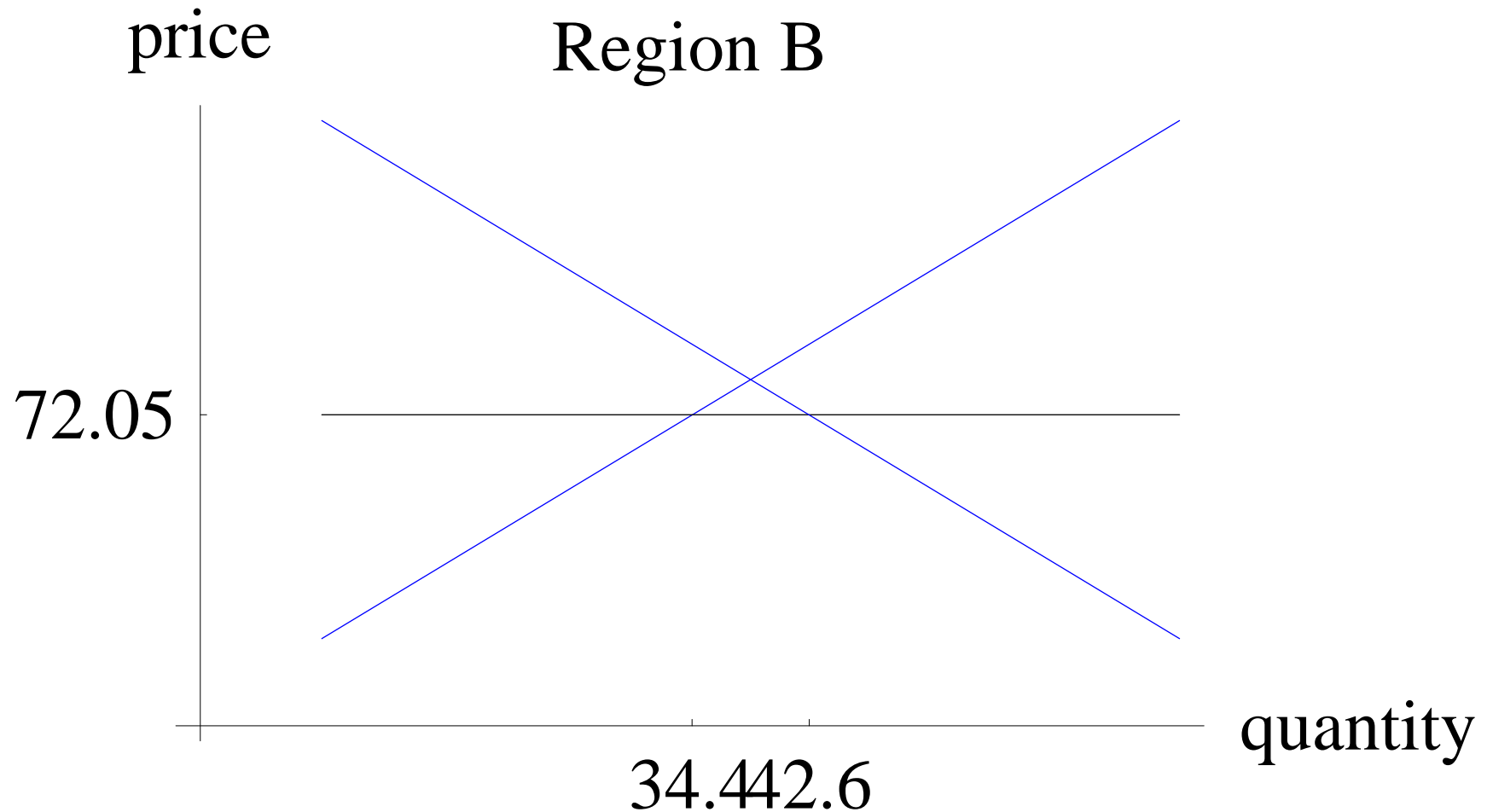
$$x_A \rightarrow \frac{281}{21}, x_B \rightarrow \frac{298}{7}, x_C \rightarrow \frac{1354}{21},$$

$$q_A \rightarrow \frac{1433}{21}, q_B \rightarrow \frac{241}{7}, q_C \rightarrow \frac{373}{21}$$

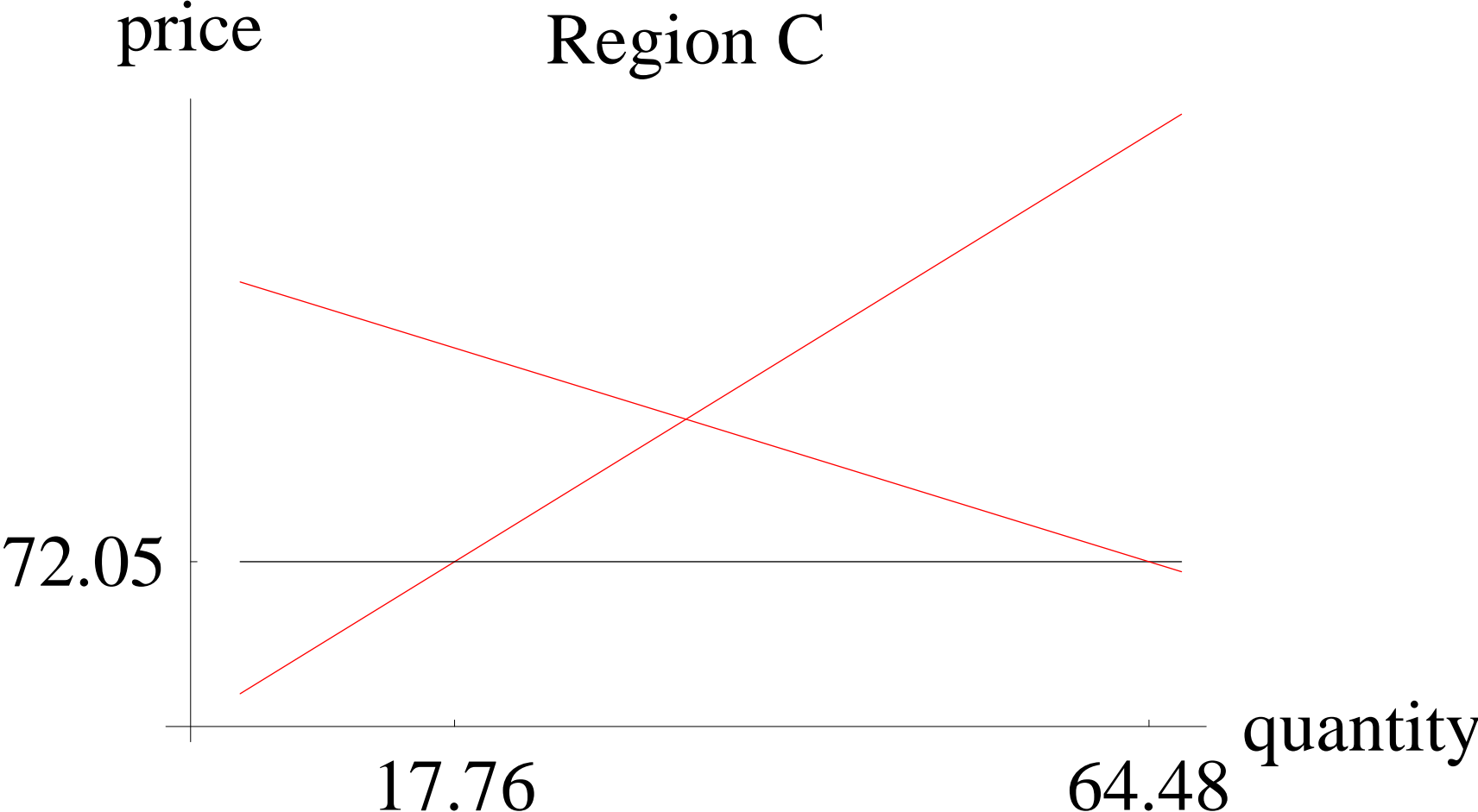
Competitive equilibrium



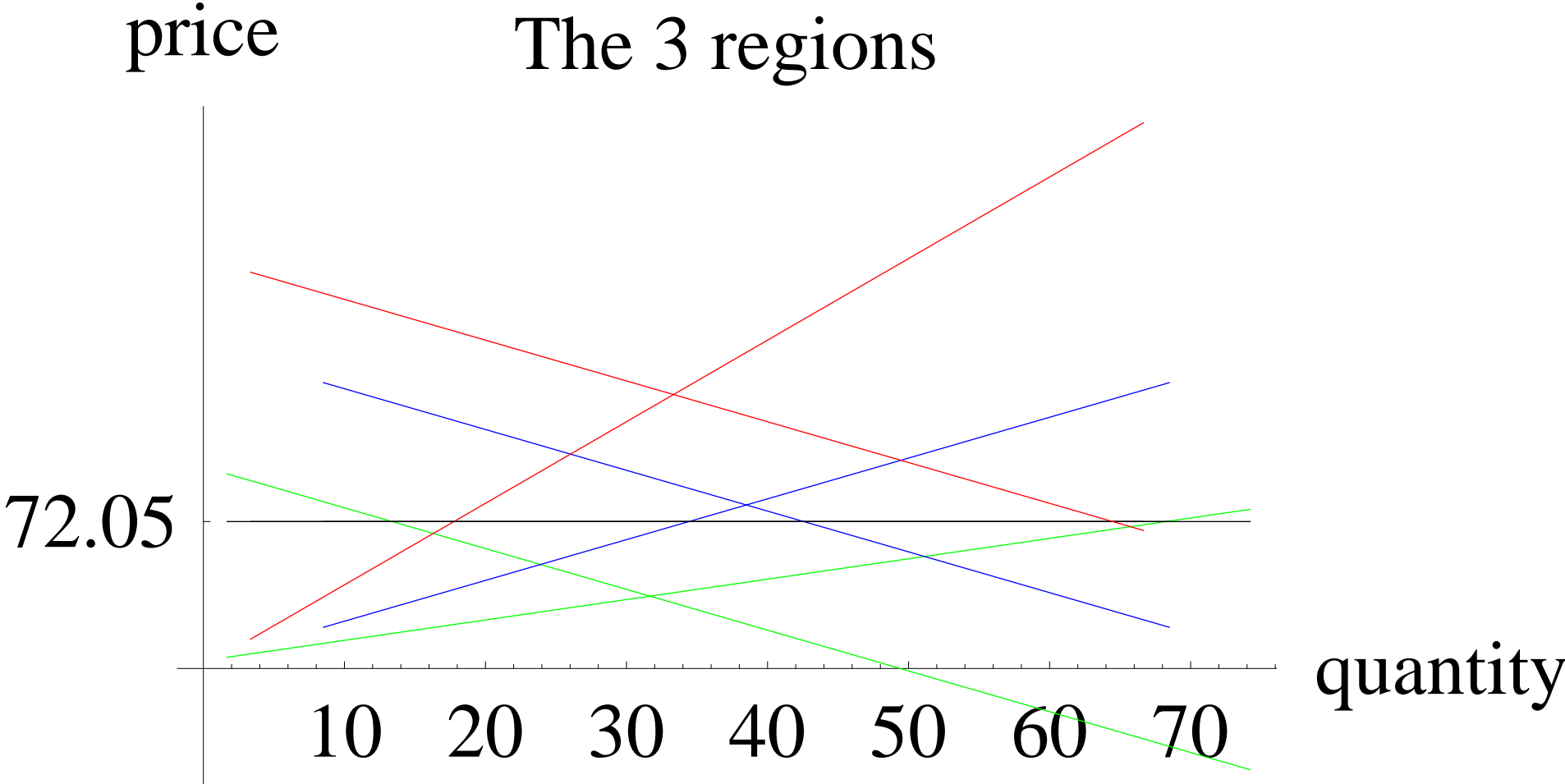
Competitive equilibrium



Competitive equilibrium



Competitive equilibrium



Competitive Equilibrium

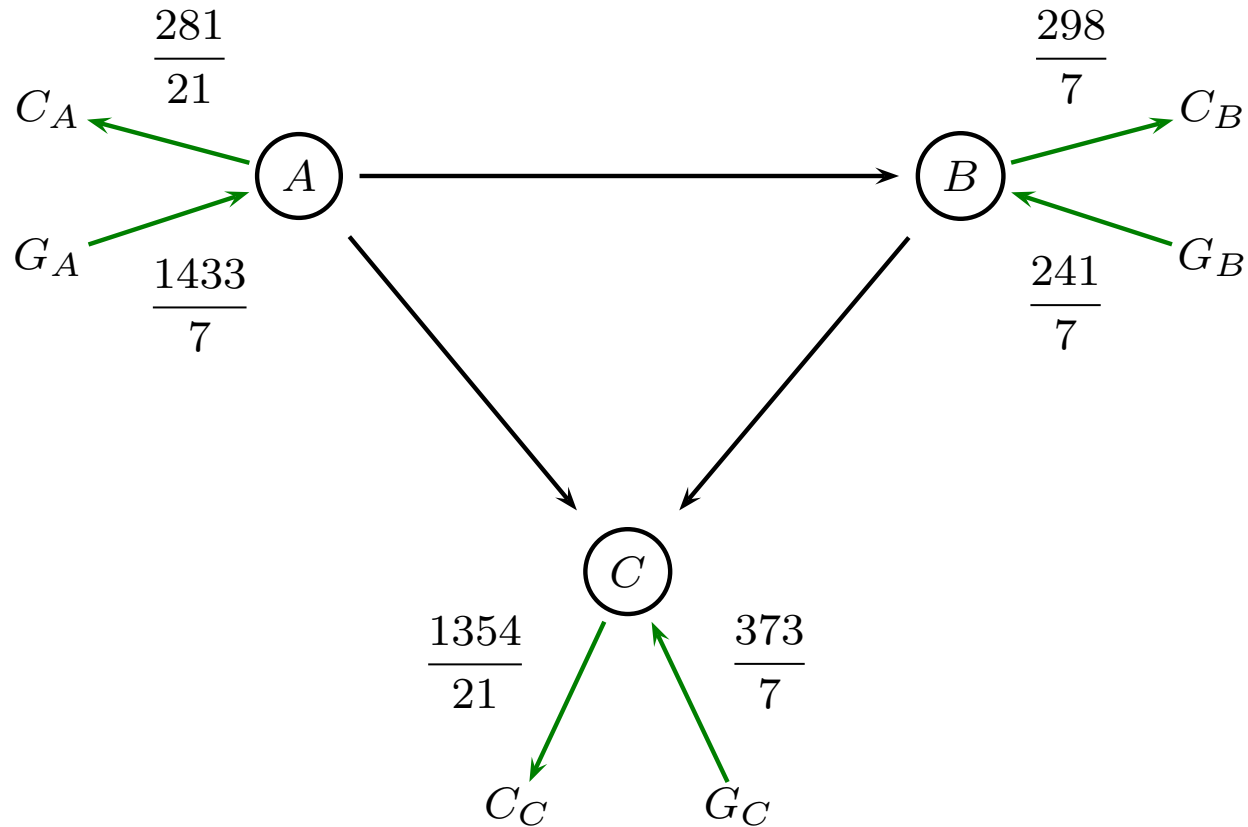
Note that in this equilibrium

- Region A *exports* $\frac{384}{7}$ MWH
- Region B *imports* $\frac{57}{7}$ MWH
- Region B *imports* $\frac{327}{7}$ MWH.

Note that the competitive equilibrium allocation is **socially optimal**.

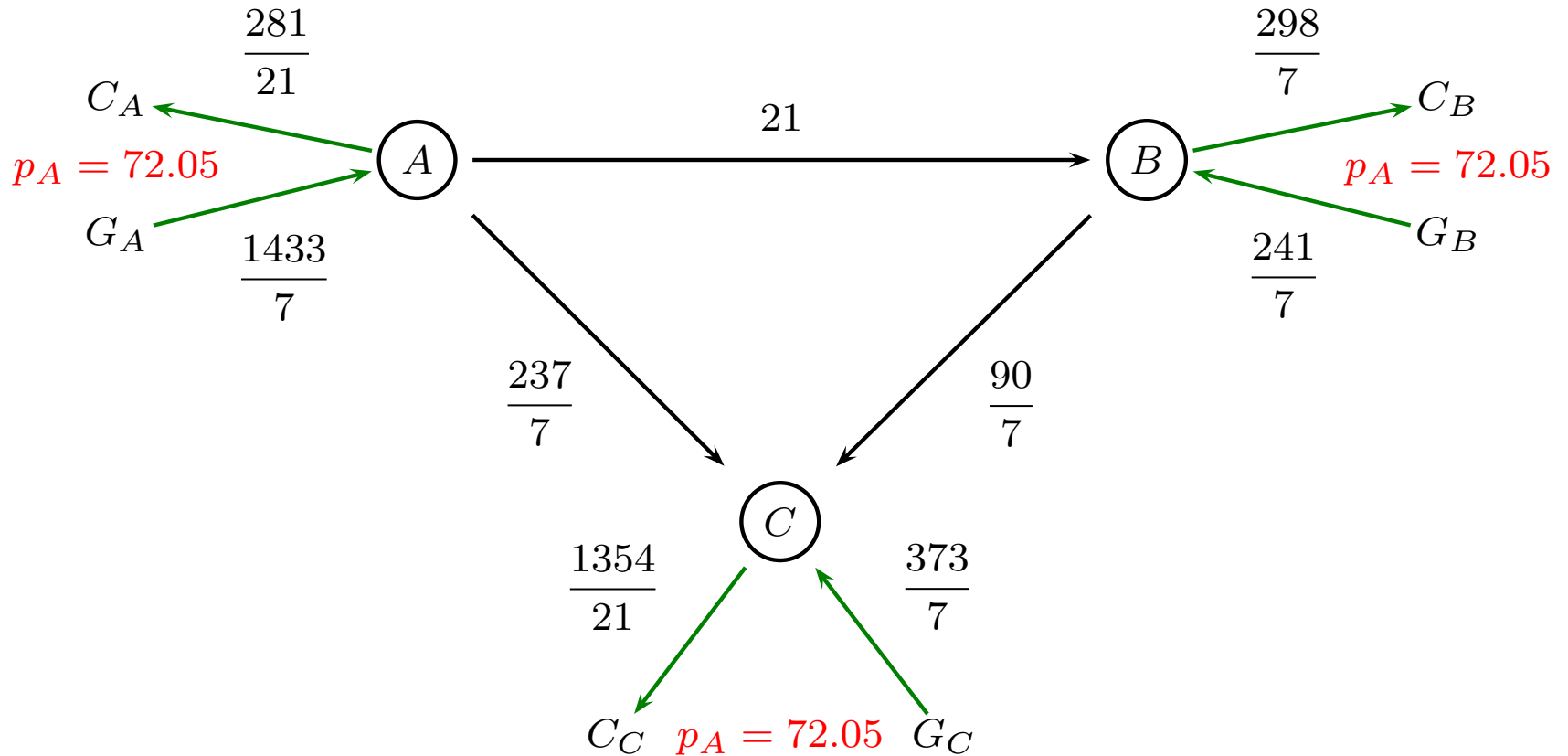
Economic dispatch

The socially optimal dispatch



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Constrained equilibrium

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Constrained equilibrium

- Assume that the maximum amount of power that can flow through the line is 16 units.
- Therefore, the above competitive equilibrium cannot be implemented because according to it, there are 21 units of power flowing along the Region A – Region B line.
- What is the socially optimal allocation of resources?
- What is the consumption and generation levels in each region that maximize the social surplus?

Social optimum

In order to find the social optimum we need to solve the following problem:

$$\begin{array}{ll} \max & \sum_{i \in \mathcal{R}} (U_i(x_i) - C_i(q_i)) \\ \text{s.t.} & \left\{ \begin{array}{l} x_A + x_B + x_C = q_A + q_B + q_C \\ \frac{1}{3}(q_A - x_A) - \frac{1}{3}(q_B - x_B) \leq 16 \end{array} \right. \end{array}$$

Social Optimum

In our case, the problem is

$$\begin{aligned} \max \quad & \left(m + \left(\frac{2075}{21} - x \right) x \right) - \left(\frac{80q}{21} + \frac{q^2}{2} \right) + \\ & \left(m + \left(\frac{3301}{21} - x \right) x \right) - \left(\frac{67q}{21} + q^2 \right) + \\ & \left(m + (201 - x) x \right) - (q + 2q^2) \\ \text{s.t.} \quad & \begin{cases} x_A + x_B + x_C = q_A + q_B + q_C \\ \frac{1}{3}(q_A - x_A) - \frac{1}{3}(q_B - x_B) \leq 16 \end{cases} \end{aligned}$$

Social optimum

The Lagrangian is

$$\begin{aligned}\mathcal{L} = & \left(m + \left(\frac{2075}{21} - x\right) x\right) - \left(\frac{80q}{21} + \frac{q^2}{2}\right) + \\ & \left(m + \left(\frac{3301}{21} - x\right) x\right) - \left(\frac{67q}{21} + q^2\right) + \\ & \left(m + (201 - x) x\right) - (q + 2q^2) - \\ & \lambda(x_A + x_B + x_C - q_A - q_B - q_C) - \\ & \mu\left(\frac{1}{3}(q_A - x_A) - \frac{1}{3}(q_B - x_B) - 16\right)\end{aligned}$$

Social Optimum

The first order conditions are:

$$\frac{2075}{21} - 2x_A - \lambda + \frac{\mu}{3} = 0$$

$$\frac{3301}{21} - 2x_B - \lambda - \frac{\mu}{3} = 0$$

$$201 - 2x_C - \lambda = 0$$

$$-\left(\frac{80}{21}\right) - q_A + \lambda - \frac{\mu}{3} = 0$$

$$-\left(\frac{67}{21}\right) - 2q_B + \lambda + \frac{\mu}{3} = 0$$

$$-1 - 4q_C + \lambda = 0$$

$$q_A + q_B + q_C - x_A - x_B - x_C = 0$$

$$16 - \frac{q_A - q_B - x_A + x_B}{3} = 0$$

Economic Dispatch

The solution to this system of equations is

$$x_A \rightarrow 16, \quad x_B \rightarrow 39, \quad x_C \rightarrow 64,$$

$$q_A \rightarrow 63, \quad q_B \rightarrow 38, \quad q_C \rightarrow 18,$$

$$\lambda \rightarrow 73, \quad \mu \rightarrow \frac{130}{7}$$

Competitive equilibrium

- **Question:** Can the above outcome be obtained as a result of decentralized trade?

Competitive equilibrium

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- **Answer:** Yes!

Competitive equilibrium

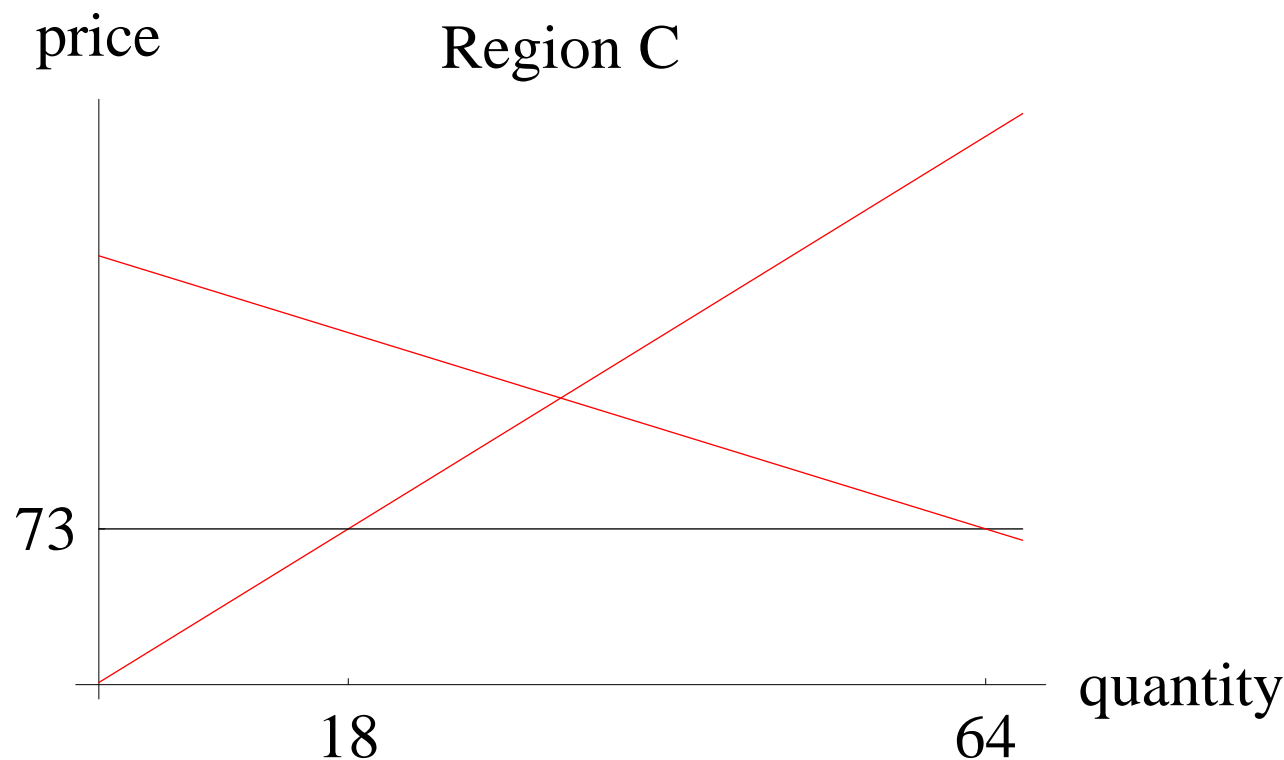
- **Question:** Can the above outcome be obtained as a result of decentralized trade?
- **Answer:** Yes!
- **Question:** How?

Competitive equilibrium

- **Question:** Can the above outcome be obtained as a result of decentralized trade?
- **Answer:** Yes!
- **Question:** How?
- **Answer:** As follows.

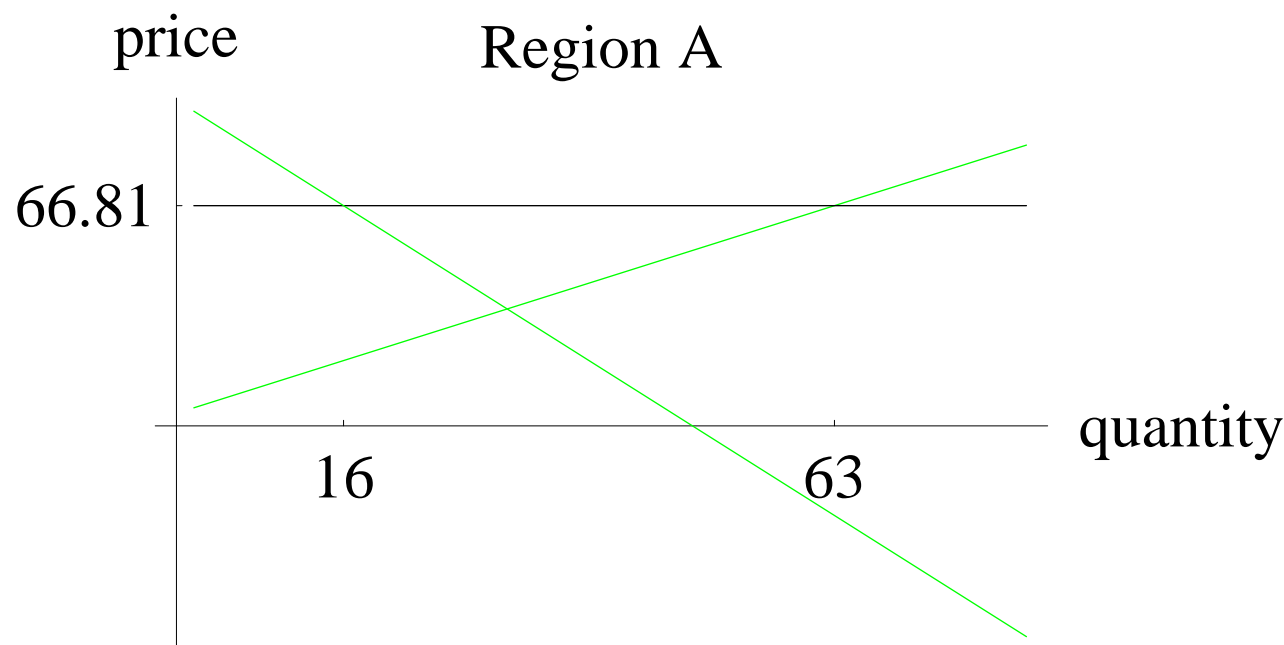
Competitive equilibrium

- What would happen in Region C if the local price of power was 73?
- To answer this question we need to look at the Region C market



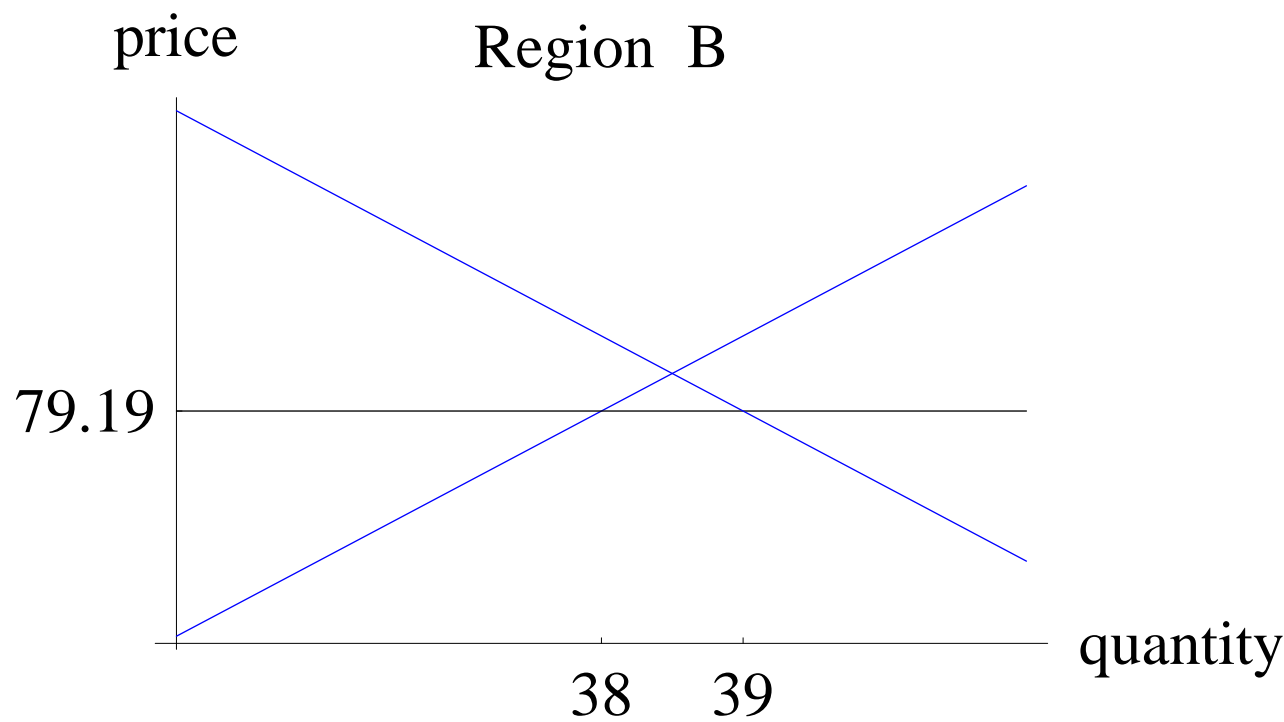
Competitive equilibrium

- What would happen in Region A if the local price of power was $73-130/7$?
- To answer this question we need to look at the Region A market



Competitive equilibrium

- What would happen in Region B if the local price of power was $73 + 130/21$?
- To answer this question we need to look at the Region B market



Competitive equilibrium

We see that if

- the price in Region A is $73 - 130/21$,
- the price in Region B is $73 + 130/21$, and
- the price in Region C is 73,
- the quantities demanded and supplied in each of the regions coincide with the socially optimal quantities.

Competitive equilibrium

In this equilibrium:

- Region A **generators** sell 63 units at $\$73-130/21$ MW
- Region A **consumers** buy 16 units at $\$73-130/21$ MW
- Region B **generators** sell 38 units at $\$73+130/21$ MW
- Region B **consumers** buy 39 units at $\$73+130/21$ MW
- Region C **generators** sell 18 units at $\$73$ MW
- Region C **consumers** buy 64 units at $\$73$ MW

Therefore

- The amount of money paid by **consumers** is \$8,829.38
- The amount of money got by **generators** is \$8,532.24
- The difference is \$297.14
- Where does this difference go?

Transmission rent

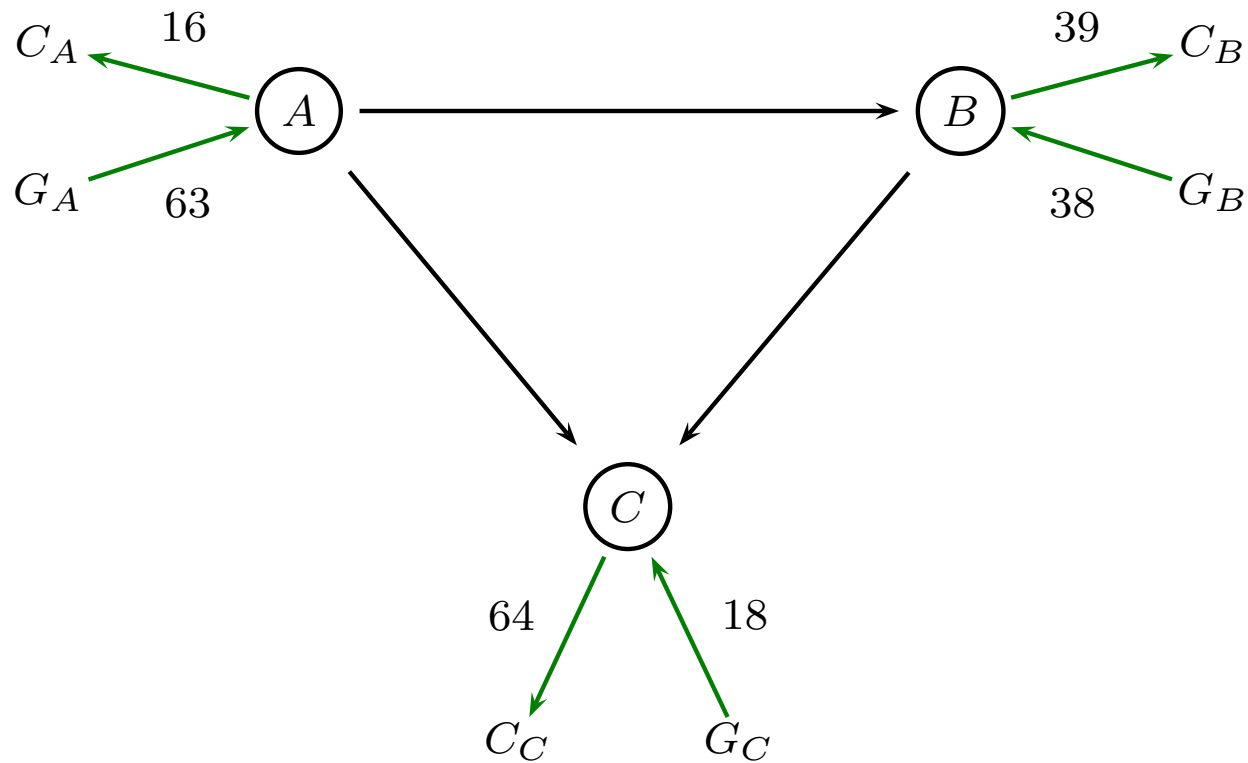
The difference goes to the transmission owners.

The transmission owners charge $\$130/7$ for each unit that transits along the line and make a revenue of

$$\$130/7 \times 16 = \$297.14$$

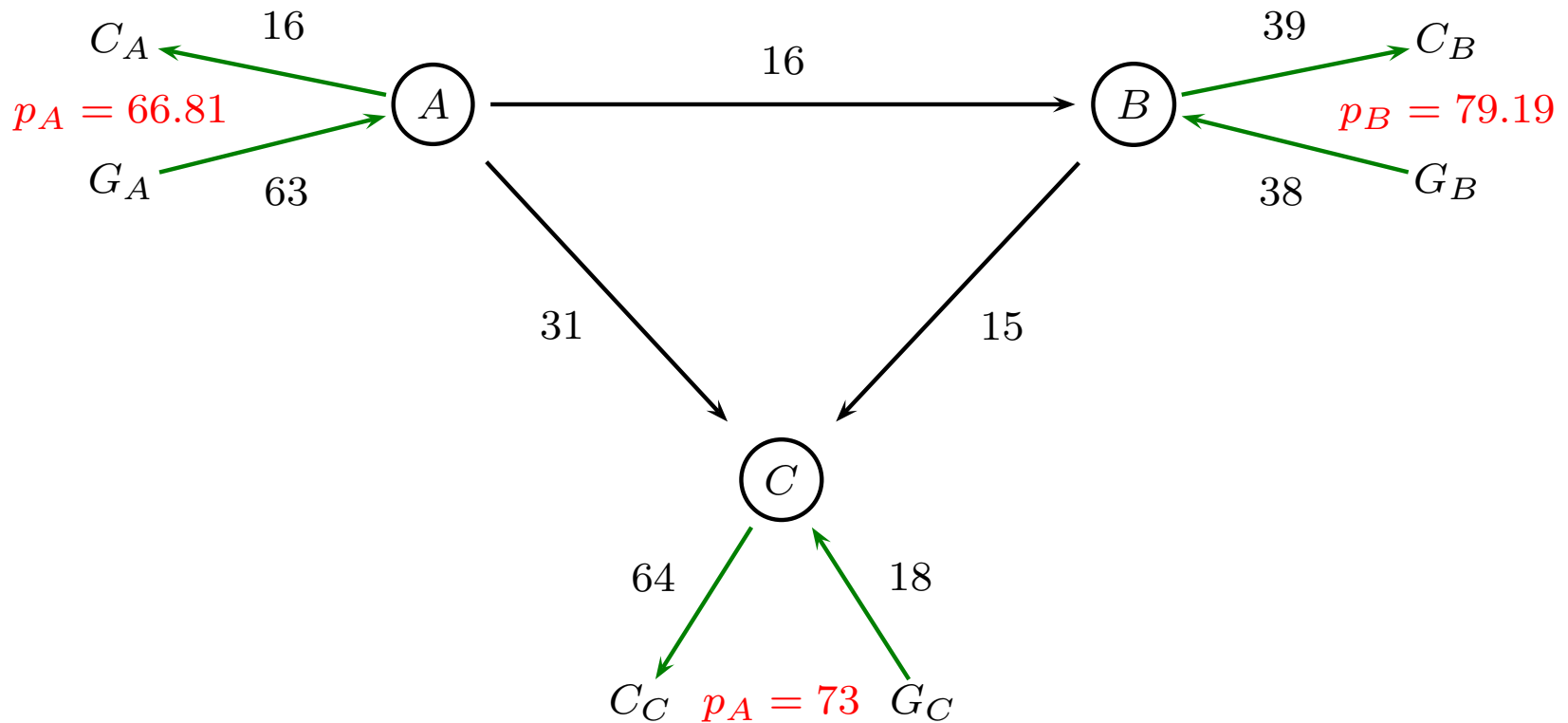
Economic dispatch

The socially optimal dispatch



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Competitive equilibrium

The *competitive equilibrium* consists of

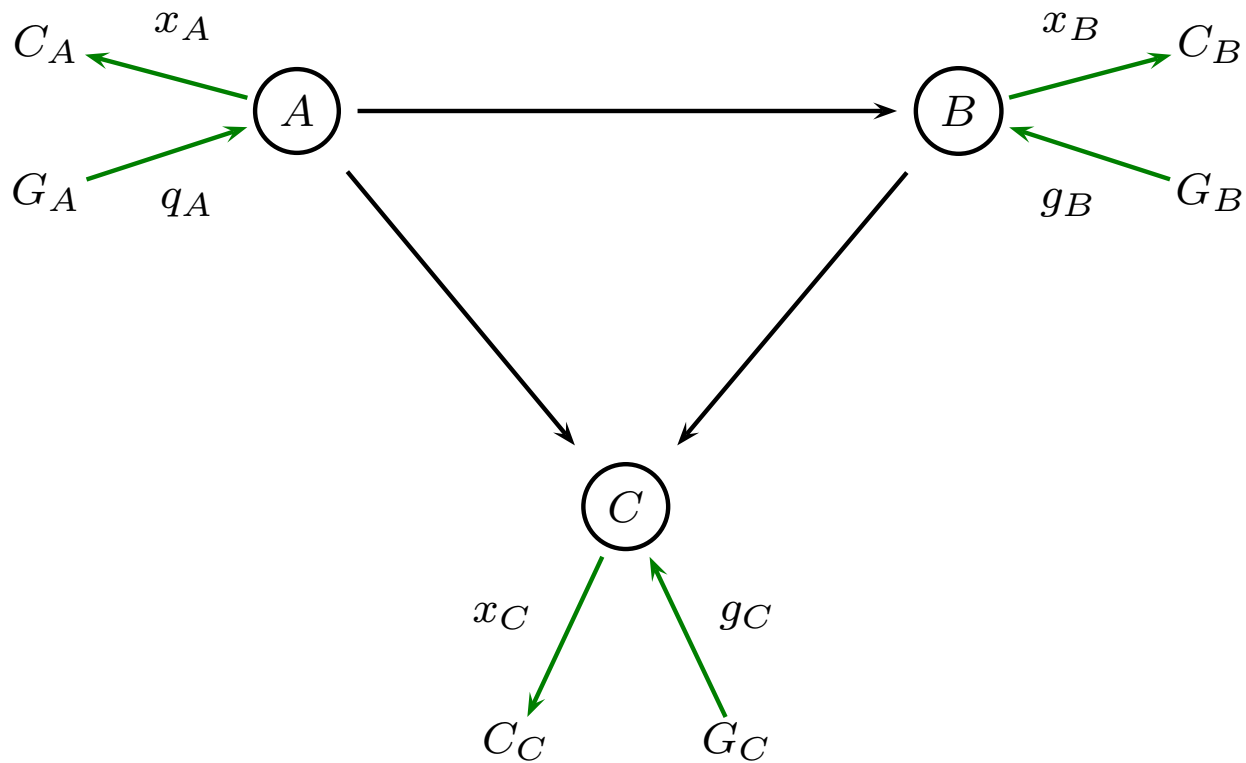
- A price $p_A = \$(73 - 130/21)$ per MW in Region A
- A price $p_B = \$(73 + 130/21)$ per MW in Region B
- A price $p_C = \$73$ per MW in Region C
- A transmission charge of $\mu = \$130/7$ per MW

such that

- $S_A(p_A) + S_B(p_B) + S_C(p_C) = D_A(p_A) + D_B(p_B) + D_C(p_C)$
- The power transmitted does not exceed the capacity of the line. In fact it equals the capacity of the line given that the transmission charge is positive
- One cannot make money by buying power from generators in any bus, transmitting it through the grid and selling it to consumers in some other bus.

Economic dispatch

The socially optimal dispatch



Economic dispatch

The socially optimal dispatch

